ALGORITHMIC CAUSAL SETS FOR A COMPUTATIONAL SPACETIME

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1. Introduction

In this paper we discuss some achievements and ongoing investigations at the intersection between two active research areas: we refer to the first one, somewhat older, broader and fuzzier, by the name 'Computational Universe Conjecture'; the second is known as 'Causal Set Program'.

1.1. Computational Universe Conjecture. This view is centered on the idea that physical phenomena are best understood in terms of digital information processing concepts. In its most extreme forms, it suggests that the universe is discrete, deterministic, finite, and evolves by simple computing rules. The central conceptual equation for this line of thought is:

\[ \text{complexity in nature} = \text{emergence in computation}. \]

We still lack experimental evidence for this conjecture, and the situation is unlikely to change in the near future, since the precise nature of the 'universal computation' is imagined to manifest itself at the tiniest spacetime scales – usually associated with the Plank units of \(10^{-35}\)m and \(10^{-44}\)sec – and much below the reach of current experimental setups. Thus, the supporting arguments are still of subjective, aesthetic or metaphorical nature.

The strongest of these consists perhaps in the widely recognized fact that simple models of computation can produce highly complex patterns, sometimes similar to those found in nature. This circumstance has been investigated and divulged, in particular, by S. Wolfram, with his extensive behavioral analysis of cellular automata and other simple models [28], and is taken by some scientists, not without considerable skepticism by others, as a valid motivation for mining the space of simple algorithms in search for the ultimate, unifying, computation-based theory of physics.

The idea of relating the dynamics of our universe to the computations of cellular automata has been pioneered by Konrad Zuse [29, 30] and Ed Fredkin [8, 16], and the general idea of a computable universe has been also investigated, under a variety of perspectives, by Lloyd [12], Schmidhuber [24], Tegmark [26], to mention a few.
1.2. Causal Set Program. The central idea of this approach to quantum gravity, pioneered by Bombelli, Meyer and Sorkin, is that, at the smallest scales, spacetime is best described in terms of the simple, discrete and flexible mathematical structure of causal sets [6, 23, 25, 7].

A causal set (or 'causet') is a finitary, partially ordered set, that is, one provided with a binary relation '≺' which is reflexive, antisymmetric and transitive, \(^1\) and such that the number of elements between any two elements is finite. Thus, a causet is conveniently represented by a graph with directed arcs and no cycles. Causet nodes represent spacetime events, the number of nodes in a subgraph measures the volume of a corresponding spacetime region, and the order relation '≺' defines the causal structure among events – a structure which, in the continuum, is usually described in terms of lightcones. This is summarized by the conceptual equation:

\[
\text{spacetime geometry} = \text{order} + \text{number}.
\]

Causets are important because the order and number information that they encode is sufficient for determining the metric tensors of General Relativity (see e.g. [22]).

1.3. The plan. By plainly merging the basic assumptions of the two discussed approaches, we come to the view that the universe is best described in terms of causal sets, and that these are of algorithmic nature.

A key objective of the Causal Set Program is to devise appropriate methods for growing causets, and in Section 2 we shall survey two major probabilistic techniques for doing so. But we are interested in replacing the probabilistic approach by a deterministic, algorithmic one. Thus, in Section 3 we introduce a general technique for representing any sequential computation, of any model, as a causal set, we show its close analogy with one of the considered probabilistic techniques, and illustrate it by an application to Turing machines.

In Section 4 we discuss 'touring ant' models of computation, focusing on 'trinet mobile automata', one in which the somewhat rigid structure of an infinite Turing machine tape is replaced by the flexible structure of a growing graph. This graph can be pictured as the external boundary of the dynamic spacetime; the computation is carried out by a stateless 'ant' that lives on it.

Both for probabilistic and for algorithmic causets, we shall focus on the quantitative, emergent feature of dimensionality, and on estimation techniques for it. On the other hand, there is little doubt that algorithmic causets outperform probabilistic ones in the variety of observed qualitative emergent properties, and in Section 5 we summarize some of them, including deterministic chaos and 'particles'. Further discussion on dimension estimators, and on their mutual (in-)compatibility, is provided in Section 6. Section 7 presents some concluding remarks.

\(^1\)Some authors, however, adopt the irreflexive convention that an element does not precede itself.
1.4. And the quantum mechanical view? Is one causet just enough to represent spacetime? Are we looking for The Causet – a unique graph structure telling us the whole history of our universe?

In quantum mechanics, the dynamics of a physical system is fully encoded by the superposition of its possible configurations, which may take the discrete form of a sum over histories or the continuum form of a path integral. In light of the excellent predictive power of quantum mechanics, most approaches to quantum gravity – for example Causal Dynamical Triangulations (CDT) [11, 13] – attempt to transpose the superposition concept to the study of spacetime, and concentrate on the search for the correct formulation of a gravitational path integral, or sum over histories, where a history would correspond to a single instance of spacetime.

The superposition of spacetime instances is regarded as an almost obligatory (albeit arduous!) approach by most researchers in quantum gravity, but, in our opinion, this is not a valid excuse for avoiding an accurate study of individual, and algorithmic (as opposed to probabilistic!) spacetime instances. Why?

With probabilistic techniques, any produced causet appears equivalent to any other, and preferring one in particular makes no sense. But when choosing a deterministic model of computation, a tiny fraction of the produced causets spring out as vastly more interesting than all the others. By 'interesting' we mean that they individually exhibit localized structures, a mix of order and disorder, self-similar patterns, and all those phenomena that are so widely investigated in [28], and that seem to manifest, more or less explicitly, also in nature. It is then reasonable to expect these special spacetime instances to play some key role in providing 'the final picture', be it based on superposition or on something else.

2. Causts from probabilistic procedures

In this section we briefly review two statistical causet construction techniques that have been investigated in the Causal Set Program.

2.1. Causts from random sprinklings. Let us consider the set $S$ of 50 points uniformly distributed in some square region of two-dimensional Euclidean space $E^2$, shown in Fig. 1-upper-left. If we interpret the vertical dimension as time $t$, and the horizontal dimension as space $x$, and if we replace the Euclidean metric by the Lorentzian pseudo-metric with signature $(+,−)$, so that the squared distance between two points $e_1(t_1, x_1)$ and $e_2(t_2, x_2)$ is given by:

$$d^2(e_1, e_2) := +(t_1 - t_2)^2 - (x_1 - x_2)^2$$
we have transformed the Euclidean space in a two-dimensional version, denoted $M^{1+1}$, of the flat, four-dimensional Minkowski spacetime $M^{1+3}$ of Special Relativity, and we can view $S$ as a set of events in it.\footnote{The Lorentz distance between two events is invariant under the Lorentz transformation, which maps spacetime coordinates between inertial frames of reference.}

Two events $e_1$ and $e_2$ are in time-like, light-like, or space-like relation when $d^2(e_1, e_2)$ is, respectively, positive, null, or negative. In the upper-middle graph of Fig. 1, each event is connected with all the events that are in time-like or light-like relation with it, i.e. all those that are on, or inside, its light cone. These edges represent the transitive relation redundantly; we can then eliminate redundancy by taking the transitive
reduction, as shown in the 'Hasse graph' of Fig. 1, upper-right graph, where only the essential pairs, called 'links', are retained.  

The lower-left diagram of Fig. 1 plots the distances between a specific point – we have chosen the one with lowest time coordinate – and all the 40 points that happen to fall in its future lightcone. The Lorentz distance, conveniently re-scaled, is compared with the graph-theoretic distance defined as the length of the longest path connecting two nodes, and their agreement is revealed. Note that any such longest path is only formed by links, thus Lorentz distance is fully coded in the Hasse graph.  

Finally, in the lower-right diagram of Fig. 1 we show that the Myrheim-Meyer dimension estimator attributes the expected 2D value to all the Alexandrov-intervals of sufficiently large volume (number of points) of a causet obtained from a 2D, 1000-point sprinkling. Let us define the two introduced concepts. 

Given two points \( x \) and \( y \) of a causet \( C \), an Alexandrov-interval, or simply interval, denoted \([x, y]\), is the finite set of points lying between \( x \) and \( y \) according to the partial order. An interval, with edges inherited from \( C \), is itself a causet. 

The Myrheim-Meyer dimension ('M-M') of a causet \( C \) with \( N \) events is obtained by considering the ratio \( r = R/(\binom{N}{2}) \) between the number \( R \) of event pairs \((x, y)\) that are actually related in the causet, i.e. those for which \( x \prec y \) or \( y \prec x \), and the maximum number of pairs that could have been related, given by the binomial coefficient \( \binom{N}{2} \). Ratio \( r \) is called the ordering fraction [17]. The ordering fraction of a causet obtained by sprinkling points into an interval of \( d \)-dimensional Minkowski space is:

\[
r = \frac{3d!(d/2)!}{2(3d/2)!},
\]

which decreases with \( d \) [15]. The possibly non integer M-M dimension \( d \) of \( C \) is then obtained from \( r \) by numerically inverting the above relation. 

The horizontal lines in the lower-right plot of Fig. 1 correspond to the ordering fraction values for \( d = 1, 2, 3, 4 \), and provide a reference for estimating the dimension of the analyzed causet intervals; each interval yields a point in the plot. 

2.2. Causets from transitive percolation dynamics. There are three reasons for considering this second, statistical causet construction technique: (i) it has been given considerable attention within the Causal Set Program, where it is regarded as 'perhaps the most obvious model of a randomly growing causet' [23]; (ii) it is

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3 The transitive reduction of a relation \( R \) is the smallest relation that admits the same transitive closure of \( R \). When \( R \) is acyclic, its transitive reduction is unique.

4 The length of the longest maximal chain between two points in a causet is indeed regarded, in the Causal Set Program, as the most natural analog for the geodesic length between two events in spacetime.
Consider the set of the first $n$ natural numbers, and let them represent the events of our causet. Then, for each pair of events $i$ and $j$, with $i < j$, create, with fixed probability $p$, a causal relation $i \prec j$: we have built a directed, acyclic graph, and a ‘raw’ percolation causet. In the sequel, we shall consistently use the terms ‘raw causet’ and ‘raw edges’ for referring to the immediate products of the considered probabilistic or algorithmic procedures, before applying transitive closure or reduction.

By construction, raw percolation causets satisfy two properties: (i) event pairs have all the same probability to be related, and (ii) causal relations are independent from one another.

Fig. 2-left shows a dense spectrum of ordering fraction values, yielding as many M-M dimension estimates, for a number of intervals from a 1000-node causet with edge probability 0.1. We have considered all intervals $[s, t]$ with $s$ ranging from 1 to 951, with step 50, for any possible $t$. Unlike the case of sprinkled causets (see Fig. 1), the detected dimension does depend here on interval volume: the larger the volume, the closer the approximation to dimension 1. The overall shape of these plots does not change when reducing edge probability: higher dimensions (lower ordering fractions) can be achieved, but these occur only for intervals of small volume.

For applications to spacetime modeling, we would most likely prefer causet construction techniques that can comfortably achieve, say, dimension 4 – if not the higher dimensions of string theory – also for large intervals.
One way to escape the dimensional collapse problem is to act on edge probability \( p \), making it variable. In Fig. 2-right we show an ordering fraction spectrum for a 1000-node percolation causet in which the probability of edge \((m,n)\) is a decreasing function \( (\log(n)/n) \) of the upper node. As in the previous case, we consider intervals \([s,t]\) with \( s = 1, 51, 101, ..., 951 \). The plot shows some stabilization at M-M dimensions higher than 1, but the process uniformity is compromised: keeping interval volume constant, the ordering fraction for interval \([s,t]\) now depends on the position of \( s \) in the range 1-1000: the lower the \( s \) index, the higher the ordering fraction. In particular, the points near the 2D gridline correspond to \( s = 1 \).

We shall see how naturally the deterministic causet construction techniques of the next section mimic the decreasing edge probability feature.

3. Causets from sequential deterministic computations

Let us now move on to a completely deterministic approach. The general idea is to obtain a causet— a discrete instance of physical spacetime—from a sequential computation, not as the final output of it, but as a direct representation of the causal relations among its events. We shall attribute to the state variables involved in the computation the role of causality mediators among events, based on the idea that an event that reads a variable is influenced by the event that has written it.

Let \( S \) be the set of events of a sequential computation \( e_1, e_2, ..., e_n, ... \), and \( X \) be the set of manipulated state variables. We start by assuming that \( X \) is static—no new variable is created as the computation proceeds—and that all these variables are initialized before event \( e_1 \). For deriving a causet, we represent the computation as the sequence \(((R_1,W_1),(R_2,W_2),...,(R_n,W_n),...))\), where \( R_i \) and \( W_i \) are the sets of variables respectively read and written by event \( e_i \). We call this an 'RW-sequence'; a causet derived from it is an 'RW-causet'. Following the mediation idea above, an RW-causet is readily built: there will be a directed edge from \( e_i \) to \( e_j \) if and only if \( W_i \cap R_j \neq \emptyset \): \( e_j \) reads at least one of the state variables written by \( e_i \), say variable \( x \). We express these facts by the notation \( e_i \xrightarrow{x} e_j \). In conclusion, the set of edges of the raw RW-causet is:

\[
E = \{(e_i, e_j) \in S^2 | \exists x \in X. e_i \xrightarrow{x} e_j\}.
\]

Note that the actual variable values play no role in the procedure. It may be helpful to picture \( E \) as partitioned into possibly overlapping subsets \( E_x \), each associated with a different variable \( x \):

\[
E = \cup_{x \in X} E_x, \quad \text{where} \quad E_x = \{(e_i, e_j) \in S^2 | e_i \xrightarrow{x} e_j\}.
\]

The idea of describing computations as nets of causally related events has been first introduced by Levin and Gacs [10], although their purpose was only to characterize computable functions; it is only by the work of Wolfram [28] that these graphs are viewed as possible spacetime instances.

The total order of computation steps does not represent physical time; the latter, as well as space, is expected to emerge from the growing structure of the causet.
3.1. A preliminary inspection of RW-causets. We would like to get some preliminary impressions on algorithmic RW-causets, without yet selecting a specific deterministic model of computation. One way to do this is to produce the sets $R_i$ and $W_i \text{ randomly}$. Let us also adopt two simplifications:

- $R_i = W_i$ for all $i$, that is, every event updates exactly the variables it reads; let $RW_i$ denote this set of variables.
- $|RW_i| = k$, for all $i$ and for some fixed $k$, that is, the number of variables manipulated by each event is fixed.

The condition for the existence of edge $(e_i, e_j)$ reduces to: $RW_i \cap RW_j \neq \emptyset$.

In Fig. 3-left we explore the performance, in terms of M-M dimension, of a generic, 1000-node randomized RW-causet as defined above, by providing the ordering fraction spectrum for causet intervals of variable volume, as done before. Not surprisingly, this spectrum is essentially equivalent to the one for percolation causets (see Fig. 2-left), since the probability of finding an edge between events $i$ and $j$ amounts, by definition, to the probability of having $RW_i \cap RW_j \neq \emptyset$, which is constant, under the adopted simplifications. Note, however, that there is a significant difference from ‘pure’ percolation causets: edge independency is now lost, since the existence of edges $(e_i, e_j)$ and $(e_i, e_k)$, with $i < j < k$, increases the chances of finding an edge $(e_j, e_k)$.

However, no reasonable definition of ‘algorithm’ can be based on the assumption of a static, or bounded set of memory locations. For example, the Turing machine – the archetypal model of computation – uses of an unbounded tape (although the

\[ (e_i, e_j) \in E_x \land (e_i, e_k) \in E_x \implies (e_j, e_k) \in E_x. \]
number of directly addressable cells must still be bounded, and conventionally reduces to one). Thus, in Fig. 3-right we explore the consequences of assuming a linearly growing set of memory locations, obtained, specifically, by letting each computation step introduce a new location, and write into it, with probability 0.4. As done before, for this 2000-step computation we have considered all intervals $[s, t]$ with $s = 1, 51, 101, ..., 1951$, for any possible $t$.

The branching structure of the spectrum reveals an arisen dependency between ordering fraction value and interval source ($s$); the phenomenon was already observed in Fig. 2-right, and similar remarks apply. More importantly, the spectrum reveals the potential of this method to achieve relatively high dimensional values.

3.2. **Forgetting non recent write operations: from fat to thin RW-causets.**

The reader may have noticed that, in defining the subset of raw causet edges $E_x$, we have retained an edge $(i, k)$ even in presence of some other edge $(j, k)$, with $i < j$; thus, an event that reads variable $x$ is influenced by all the events that have written $x$ earlier in the computation. This convention was suggested by analogy with the percolation technique, but it conflicts with the usually assumed overriding nature of the write operation.

So we shall also consider RW-causets in which an event that reads variable $x$ is influenced only by its most recent writer event. In this case, the set of edges contributed by variable $x$ becomes:

$$E_x = \{ (e_i, e_j) \in S^2 | e_i \xrightarrow{x} e_j \land \neg \exists k. (i < k < j \land e_k \xrightarrow{x} e_j) \}.$$

It is easy to realize that, in switching from the original, 'fat' causets, to these new 'thin' graphs, we are filtering out a large number of raw edges. Nevertheless, as long as we keep the convention $R_i = W_i$, the introduction of this more natural treatment of write operations has no effect on the M-M dimension of the causet! (The simple proof of this apparently surprising fact is omitted for space reasons.)

3.3. **The case of elementary Turing machines.** For a more concrete illustration of RW-causets, let us now show some of those that derive from the most widely known sequential, deterministic model of computation.

In an elementary Turing machine, a 2-state control head moves up and down a binary tape containing symbols from alphabet $\{0, 1\}$. The behavior of the head is defined by one of 4096 possible $2 \times 2$ state transition tables: rows are labeled by the control head states $s_1$ and $s_2$, columns by tape alphabet symbols, and entries are triples of the form $(s', b', d)$. If the control head is in state $s$ and reads bit $b$ from the cell where it is positioned, then, following what is specified in the $(s, b)$-entry of the transition table, it changes its own state to $s'$, writes bit $b'$ in the same cell,
and moves left, if $d = -1$, or right, if $d = 1$. Hence, two variables are read at each computation step – the control head state and one tape cell – and the same two components are written: using previously introduced notation, we have $R_i = W_i$, and $|RW_i| = 2$.

By proceeding via RW-sequences as described above, in [4] we have computed all the raw (and 'thin') RW-causets for the computations, of arbitrary lengths, of all 4096 machines, assuming an initial tape configuration of all 0’s, and we have shown that the derived graphs are all planar and fall into only three categories, sampled in Fig. 4: 1D, 2D flat, and curved. Beyond what is obviously suggested by visual appearance, classification is based on node-shell-growth analysis, a technique to be discussed later.

4. Causets from touring ants

We cannot expect to be able to describe physical spacetime purely in terms of the very regular RW-causets from elementary Turing machines (Fig. 4). Thus, in search for more interesting RW-causets, we move to another model of computation, based on a stateless touring ant idea. We find it convenient, and also appropriate for the present celebratory volume, to introduce this model by two progressive modifications of a standard Turing machine.

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8 For our purposes we do not need to consider termination conditions.

9 Coincidentally, these machines are provably not Turing-complete: we need more states or alphabet symbols to achieve computational universality.
4.1. Touring ants on circular binary tapes. The first modification consists in replacing the infinite, linear, binary tape with a finite, circular one, while turning the finite-state Turing machine head into a stateless, touring ant. Initially the tape has only two cells, but the ant can create new ones. At each step the ant is positioned on two adjacent cells, $c_1$ and $c_2$, that it reads. Depending on these two bits: (i) it writes $c_1$ and either $c_2$ or a new cell created between $c_1$ and $c_2$; (ii) it either remains where it was – on $c_1$ and the next cell – or it moves one step to the right.\footnote{There are $2^{16} = 65536$ possible behaviors of this type.}

Our interest for this model is due to the fact that we can picture the growing tape as an elastic circular band (Fig. 5) with black and white segments – the bits manipulated by the ant – that also represents the external boundary of the growing causet/universe. Thus, the computation takes place only at the border of the universe, locally, while spacetime arises by internal accumulation of events, and records the history of what has happened at the surface. The process is depicted in Fig. 5 but, before describing it in detail, let us reassure the reader that we are doing nothing but applying the RW-causet construction technique of Section 3. In essence, we represent the computation as a sequence $((R_1, W_1), (R_2, W_2), \ldots, (R_n, W_n), \ldots)$, where $R_i$ and $W_i$ denote, respectively, the pairs of cells read and written by event.
Then we let the cells play their role as causality mediators, using the 'thin causet' convention described in Subsection 3.2.

In Fig. 5-left we show a 10-event causet for a computation of an instance of this model (instance n. 31153, according to our numbering scheme), and the 5-cell border that these events have built. Also shown is the rule describing the ant behavior, which consists of four cases, one for each configuration of the two cells being read: in the first case, a new cell is introduced (and written), while in the remaining three cases the pairs of read and written cells coincide.

The diagram in Fig. 5-left also illustrates the use of wiggling arrows as an aid for building the RW-causet incrementally. Each one of these special arrows temporarily connects a cell to the event that has written it most recently. Consider the situation immediately before event 10: the ant is positioned on adjacent cells $x(0)$ and $y(0)$, with bits in parentheses denoting cell content. The first rule case applies, which involves the creation of a new cell. When the event occurs, we use the wiggling arrows currently pointing to the cells being read, namely arrows $9 \leadsto x(0)$ and $7 \leadsto y(0)$, for tracking back the most recent writers of those cells, namely events 9 and 7, and for creating two new causet edges $9 \rightarrow 10$ and $7 \rightarrow 10$: these reflect the causality mediation played by the two cells. Two new wiggling arrows are also added, for future use, from event 10 to the two cells it writes: $10 \leadsto x'(0)$ and $10 \leadsto new(1)$. Note that wiggling arrow $9 \leadsto x(0)$ is also dotted, for representing its disappearance, since it is now superseded by arrow $10 \leadsto x'(0)$; but arrow $7 \leadsto y(0)$ is not, since event 7 is still the most recent writer of cell $y$. Finally, in Fig. 5-right we show the causet for a 500-hundred step computation of the machine. The causet exhibits spiraling growth, with edges oriented outwards, either radially or along the spiral. Incidentally, the growth process appears irregular, as the reader may check by inspecting, for example, the lengths of the spiraling runs of adjacent square faces.

4.2. Touring ants on planar trivalent networks. The step to the next model – trinet mobile automata – is relatively short: in place of a circular tape, we now want the border of the growing causet – the memory support where the ant lives and operates – to be a 'trinet': this is our short name for a planar, trivalent, undirected graph, that is, one with undirected edges that can be drawn on a sphere while avoiding crossings, and such that each node has exactly three outgoing edges.

Note that even the circular tape could be seen as a graph – a bivalent one in which each node, corresponding to a cell, has two neighbors. However, bivalent graphs provide poor structures – just rings, thus we had to introduce binary node labels in order to obtain non-trivial behavior. On the contrary, a trinet partitions the embedding sphere into faces, or $n$-ary polygons, and this $n$-arity information is sufficient for developing interesting algorithms, without need for extra structure.
Figure 6. RW-causet construction for a touring ant on a planar, trivalent graph, appearing as the external boundary of the growing causet (left). The two rewrite rules between which the ant chooses at each step, and a specific choice for the ant moves (right). Each trinet face is hit by a temporary wiggling arrow, starting from the causet event that has updated it most recently (not all such arrows are shown).

The general operation of the trinet mobile automata introduced in [2, 3], and the application of the RW-causet construction technique to them, are simultaneously depicted in Fig. 6, which bears similarities with Fig. 5; the picture shows a trinet, the ant operating on it, and the causet growing inside. At each step the ant inspects a small portion of the graph, modifies it by one of two possible graph rewrite rules, and moves to a nearby location. The 2D Pachner rules that we use, sometimes called Expand and Exchange, are also applied in Loop Quantum Gravity (see, e.g., [14]), and are shown in Fig. 6-right. Note that, in analogy with the previous model, by rule Expand the ant adds an element to the memory support, namely a trinet face. Both the rewrite rule and the next ant location are chosen by some deterministic criterion: a specific choice of ant moves is actually made in Fig. 6, separately for each rule, while the rule choice criterion is left unspecified.

A first variant of the model [3], called three-connectivity preserving, is as follows:
(1) Start with a trinet consisting of 2 nodes connected by 3 parallel edges. 

(2) Choose rule *Exchange* whenever it does not violate three-connectivity, otherwise choose *Expand*.

(3) Move the ant to a new nearby location, only depending on the applied rule.

In a second variant [2], qualified as threshold-based, point (2) is modified as follows: choose *Exchange* whenever it does not create trinet faces with less than $k$ sides, for some predefined $k$, otherwise choose *Expand*.

Let us turn to causet construction. When applying a graph rewrite rule, the ant reads and writes trinet faces: rule *Expand* reads three faces $(A,B,C)$ and writes four $(A',B',C',\text{new})$, while rule *Exchange* reads and writes the same four faces $(A,B,C,D)$. Thus, polygonal trinet faces are an obvious choice for causality mediators among events. We can then build the RW-sequence, where the $R_i$ and $W_i$ are now sets of trinet faces, and derive causet arcs from it, as described before.

Similar to Fig. 5, Fig. 6 also shows the use of wiggling arrows for carrying out the causet construction incrementally. Event $n$ corresponds to the application of the *Exchange* rule, that modifies faces $(A,B,C,D)$. Before its occurrence, the most recent writers of faces $(A,B)$ and $(C,D)$ were, respectively, events $k$ and $h$, as shown by the wiggling arrows $k \leadsto A$, $k \leadsto B$, $h \leadsto C$, $h \leadsto D$. When event $n$ occurs, all four wiggling edges are dropped, and two new causet edges are created: $k \rightarrow n$ and $h \rightarrow n$. In addition, four new wiggling edges are created from $n$ to the new configurations of faces $A,B,C,D$.

In the next section we shall provide some experimental evidence that, in terms of emergent properties, RW-causets from touring ant models are not only much richer, expectedly, than those from elementary Turing machines, but also outperform, w.r.t. qualitative properties, causets obtained from probabilistic techniques.

5. **Emergent properties of touring ant causets**

A distinguishing feature of our universe – one which seems to play a key role also in art – is the mix of order and disorder. This feature is so obvious and pervasive that it goes almost completely unnoticed, and fails to qualify for serious physical explanation. But, if a unified, computational, causet-based theory of physics is ultimately found, we must expect it to account both for regular and irregular behaviors, and for their appropriate interplay. Order is obviously achieved with algorithmic causets, and Fig. 4 provided some examples; what about disorder? 

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11This is the smallest possible three-connected graph. A connected graph is $n$-connected when $n$ is the smallest number of edges one has to remove for disconnecting it.

12For our purposes here, we are just satisfied by defining ‘ordered’ (or ‘regular’) a behavior whose evolution can be easily predicted by visual inspection.
5.1. **Deterministic chaos in trinet mobile automata.** The ordered/disordered nature of a sequential computation is well reflected in the corresponding causet graph. But in the case of trinet mobile automata, we can conveniently use a more abstract type of plot for this purpose.

In Fig. 7 we plot the ant motion on the growing trinet, for six instances of the threshold-based model, each labeled by three parameters: the first parameter is the threshold value, and the others identify the ant moves associated, respectively, with rule *Expand* and *Exchange* (see [2] for details). More precisely, the ant ‘trajectory’ is rendered by plotting the identifier of the edge where the ant is located as a function of the computation step (edges are numbered progressively, as they are introduced, three at a time, by rule *Expand*).

The upper five diagrams in Fig. 7 summarize the typical dynamics that can be observed, with minor variants, for all parameter settings. In all cases, except for the second one, the trinet grows unbounded. The case with threshold $= \infty$ is known as
the 'fractal sequence'. All five cases appear regular, that is, completely predictable in their evolution; the fifth one stabilizes after an initial, chaotic transient.

Then, out of a few thousand instances we have inspected, we find only two exceptional (and similar) cases of deterministic chaos, or pseudo-randomness, that form a tiny class in themselves: automata (4, 17, 8) and (5, 9, 8). The dynamics of the first case is illustrated in Fig. 7-bottom, which refers to a 20,000-step computation. We have actually reached one billion \(10^9\) steps without observing any sign of stabilization! The plot also includes a fitting function, showing that the trinet growth rate is \(O(\sqrt{\text{steps}})\).

An interesting phenomenon is visible, in the diagram, particularly between steps 4000 and 5000: the ant is temporarily confined inside some region of the causet, and keeps visiting the same edges for a while. In [5] we call this feature 'causet compartmentation', show its impact on the causet graph structure – the creation of a 'hole' – and argue that: (i) these compartments may represent a first, rudimentary form of self-organization of the causet into regions that achieve partial independence from one another; (ii) they can not be expected to emerge in 'genuinely random' causets.

5.2. **Dimensional analysis.** By the node-shell-growth analysis, we identify the sets of nodes (the 'shells') at progressive distance \(r\) from a given node \(n\), and attempt to fit their growth rate by some function, in particular a polynomial or exponential. If shell sizes grow like \(r^d\), where \(d\) can be non-integer, we assign to the causet a 'node-shell-growth' dimension \(d + 1\), relative to node \(n\) (a more elaborate definition is provided in [18], under the name 'internal scaling dimension').

In general, the node-shell growth rate depends on the reference node; when computed relative to the root, the estimate reveals some global feature of the growing spacetime which must not necessarily be confirmed by localized observations from the inside. For example, we find regular causets from Turing machines that appear 3D from the root, and 2D from a generic internal node (see [4], Fig. 11). Fig. 8 illustrates a fairly regular causet from a threshold-based trinet mobile automaton. Node-shell growth from the root is approximately quadratic, yielding a 3D estimate, and, in this case, the estimate is roughly confirmed by the localized views from the other, 'internal' nodes.

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13In [4], we have taken exponential shell growth as an indication of negative (hyperbolic) curvature. The obvious weakness of this approach is that fails to treat dimension and curvature separately, and to attribute a finite dimension to a curved causet. Decoupling causet dimension and curvature is a crucial, but still largely unsolved problem.

14Obtaining a 3D estimate for a causet deriving from a computation on a 1D support - the Turing machine tape - is interesting: it proves that the causet construction process is not always incrementing by at most one unit - the time component - the dimension of the underlying support.
Figure 8. Causet from a 3000-step computation of a trinet mobile automaton – code (5, 17, 2) (left and center). Node-shell growth from the root is approximately quadratic, yielding a 3D estimate (right).

Figure 9. Causet from a 30,000-step computation of a trinet mobile automaton – code (5, 9, 8) (upper). Node-shell growth from node 1 appears irregular (solid line, lower-left plot), while from node 12,000 it is approximately quadratic (dotted lines).

In Fig. 9 we show the causet for one of the pseudo-random computations mentioned in Subsection 5.1. Node-shell growth from the root looks rather irregular; from internal nodes, however, the growth appears approximately quadratic, yielding a local 3D estimate.
FIGURE 10. Particles in regular causets derived from: 3000-step computation of a 2D Turing machine (left); 4000-step computations of trinet mobile automata (4, 16, 2) (center) and (6, 10, 2) (right).

5.3. **Particles?** In the context of the Computational Universe Conjecture, the term ‘particle’ immediately evokes the ‘gliders’ and ‘spaceships’ of Conway’s Game of Life – a two-dimensional cellular automaton – or the interacting, localized structures of Wolfram’s one dimensional, Elementary Cellular Automaton (ECA) 110. These phenomena are characterized by a remarkable mix of order and disorder: in ECA 110, for example, the background is completely regular, but the overall interaction pattern of particle trajectories and collisions appears pseudorandom [28].

In cellular automata all cells are updated simultaneously; but we are interested in obtaining interacting particles from touring ant models, in which the privilege of parallel operation is not given, and this seems to be harder. A sample of what can be currently achieved is provided by the regular causets in Fig. 10. The causet on the left comes from the computation of a 2D Turing machine, or ‘turmite’ [19], which operates on a square grid (for an interactive demonstration, see [20], turmite n. 4). The remaining two causets derive from trinet mobile automata. In all three cases, particles emanate, radially or in spirals, from the root, moving on a periodic background, as in ECA 110. But we are still far from the complexity of interactions of the latter: the only interaction observed in Fig. 10 is the deflection occurring when the spiraling and radial particles collide, in the first causet.

6. **More on dimensional analysis**

We have applied the ‘Myrheim-Meyer’ estimation technique to stochastic causets, and the node-shell-growth technique to algorithmic causets. Shouldn’t we rather apply indifferently either technique to either causet type?

First, imagine to apply the M-M dimension estimator to the causets from Turing machines and trinet mobile automata. The result is readily anticipated: no matter
how complex they appear, all these causets will exhibit a disappointing M-M dimension 1! The reason is simple. Consider Turing machines (the case of trinet automata is analogous). One of the causality mediators is the control head state $s$, which plays its role at each step; thus, the subset $E_s$ of raw edges mediated by $s$ forms a path that traverses all the events of the computation, turning the raw causet into a totally ordered one. Then, transitive closure creates an edge for every node pair, thus obscuring the potentially interesting structure of the raw edges deriving from tape cell mediation. In conclusion, the ordering fraction of any causet interval is 1, and so is the M-M dimension. In this case, only the raw causet retains non-trivial information, and the node-shell-growth estimator appears to be the right choice for detecting it.

Conversely, imagine to apply node-shell-growth analysis to causets obtained by sprinkling (Subsect. 2.1). In this case, there is no difference between the raw and the transitively closed causet, since, by definition, all causal relations are explicitly included in the raw graph. The analysis is still possible, but makes little sense, since, starting from node $s$, we end up with just one gigantic shell at distance 1 from it, containing all nodes in its future light cone.

We have considered two perhaps extreme cases, which trigger a variety of questions: on the relevance of totally-ordered causets for applications to spacetime modeling, on the choice among causet forms, among dimension estimation techniques, on their agreement, and on the inter-dependence of these choices.

The question on totally-ordered causets goes beyond mere dimensionality concerns. Nevertheless, let us mention one reason for not excluding these causets from the agenda. Although the potentially interesting structure of a totally-ordered causet is washed away when transitive reduction is applied globally, it may still survive when the operation is applied locally. In Fig. 11-left we identify two regions of the totally ordered, second causet of Fig. 10. The central diagrams shows the devastating effect of applying transitive reduction to the first region or, equivalently, to the whole causet. On the contrary, the r.h.s. diagram shows that the particle structure is preserved, although modified, when transitive reduction is applied, locally, to a peripheral region.

How about the choice of causet form: raw, transitively closed, or Hasse?

Attributing importance to raw causets means attributing physical relevance to the 'spurious' edges of the causet graph – those that disappear with (global) transitive reduction. There might be deeper reasons for preserving spurious edges, beside the practical fact that they tolerate local transitive reduction. 15 And yet, for simplifying

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15 Referring to Turing machines, for example, one reason could be that the spurious causet edges introduced by the mediation of tape cells are the only vehicle for letting information on the underlying tape topology pop up to the causet level.
our discussion, let us now assume that we discard raw causets, in favor of the other two, somewhat 'cleaner' forms.

Quite obviously, a transitively closed causet and its transitive reduction – the Hasse form – contain the same information, being in one-to-one correspondence with each other: nothing fundamental is involved in the choice between them. In principle, node-shell-growth analysis can be applied to both forms, providing different outcomes; but we have seen, by the example of sprinkled causets, that it is a bad idea to apply this technique to the transitively closed form. As a second example, consider a regular square grid in which vertical and horizontal edges point, respectively, upwards and to the right. In this case, node-shell sizes from a generic node grow linearly with distance, yielding the expected 2D estimate; but if we apply node-shell analysis to the transitive closure, the already mentioned shell collapse occurs.

Note that the M-M technique agrees on the 2D estimate for the 2D regular grid. We may then be tempted to declare that both estimation techniques are valid, interchangeably, provided the causet is presented in adequate form. But, can we always expect agreement between the M-M estimate, for the transitively closed causet, and the node-shell-growth estimate, for its transitive reduction?

We can test this conjecture by analyzing node-shell-growth for the transitive reduction of a causet obtained by sprinkling in a 2D manifold. In our simulations we could compute only a few node-shells, due to the computational cost of transitive reduction; these are insufficient for a reliable fitting of the growth rate, although
the apparent trend is markedly above the 2D estimate. However, visualizing the structure of these few shells is useful. Fig. 12 refers to a Hasse causet obtained by sprinkling 1000 points in a 2D, diamond-shaped region. In the six copies of the graph we have highlighted the node-shells at progressive distance – 1 to 6 – from the bottom node. The diagrams reveal the highly non-local nature of sprinkled causets, which reflects the non-locality of Minkowski space. It is doubtful that these features be compatible with a sensible application of the node-shell-growth technique.

We have only scratched the surface of a rather intricate area – dimension estimation for causal sets – in which many questions are still open. No single estimator has yet emerged as the ideal option for causet analysis. It has been often observed that an estimate becomes reliable when two or more techniques agree on it; this does not always happen, and adding node-shell-growth analysis to the family seems to bring further complication.

\footnote{This non-locality feature is ultimately responsible for the counterintuitive phenomena of Special Relativity, such as the twin paradox: one twin can travel between spacetime points \( s \) and \( t \) along a long geodesic path, while his brother can take a much shorter (but accelerated) path between the same points, thus experiencing a shorter time delay.}
7. Conclusions

In this paper we have suggested that interesting contributions to a theory of space-time – which one might boldly equate to a theory of everything – can be provided by the study of the emergent properties of algorithmic causal sets, intended as representations of the causal relations among the events of some sequential, deterministic computation. In which model of computation?

In computability theory, all Turing-complete models – those that can simulate a universal Turing machine – are equivalent. Should we regard them as equivalent also with respect to spacetime theory? Fredkin [9] refers to this problem as ’the tyranny of computational universality’, and suggests a choice criterion: there should be a one-to-one mapping between the states and function of the real world and those of the model. This leads him to focus on second-order, reversible, universal cellular automata (RUCA), and the SALT model [16].

Under our causet-oriented perspective, a most direct reformulation of Fredkin’s criterion would consist in requiring the algorithmic causet and physical spacetime to be isomorphic. By following this (largely speculative) track, one is led to compare the causet types corresponding to the various models. Our experiments [4, 5] have shown that, while the simplest patterns, e.g. polymer-like or hyperbolic, are pervasive in all causet classes, fundamental differences remain. Discriminating factors include properties such as planarity, node degree unboundedness, and total order, which may or may not be satisfied [4]. The conclusion is that universal models of computation are not equivalent, with respect to spacetime modeling, and we are still left with an open choice problem.

In spite of this problem, in this paper we have expressed some preference for the class of ‘touring ant’ models. In doing so, we have excluded cellular automata, possibly the predominant model of the Computational Universe Conjecture. 17 Our preference for trinet mobile automata, beside the aesthetic appeal of a Cosmos run by a single, memoryless ant, comes from the high degree of abstraction and flexibility offered by graphs and graph rewriting. For example, when the graph is planar and trivalent (a ‘trinet’), by just applying the introduced Expand and Exchange rules, we can create new triangular faces, grow them to become \(n\)-polygons, for any \(n\), and freely move them around, bringing any face in contact with any other, thus turning the dynamic graph into a lively population of entities – the faces (see [21] for a demonstration). Dynamic trinets appear as an ideal stage for the emergence of complex behavior, although detecting it may be harder than with cellular automata.

\[\text{\footnotesize{17Perhaps counterintuitively, both the parallel operation of cellular automata and the sequential, localized operation of touring ants, can \textquoteleft implement\textquoteright{} the multiplicity of concurrent activities that we expect to observe in a realistic model of our universe, as discussed in [28], at p. 487.}}\]
Some readers may have spotted, in our picture of trinets as boundaries of a growing causet/spacetime, a vague analogy with the holographic principle proposed by 't Hooft and Susskind. In its most general form, this principle suggests that the universe could entirely depend on a two-dimensional information structure found at the cosmological horizon [1]. One would then be interested in entropy measures for the boundary, and this is another aspect where trivalent graphs, or their duals (triangulations), offer advantages. Tutte [27] establishes that the number of distinct planar triangulations with \( n \) nodes, i.e., of trinets with \( n \) faces, is
\[
\psi_n = \frac{2(4n+1)!}{(3n+2)!(n+1)!}.
\]
The amount of information carried by a trinet, in bits, can be expressed by \( \log_2 \psi_n \), which grows linearly with \( n \) and yields an estimate of 3.2451 bits/face. Calculations of this type could help in establishing possible correspondences between algorithmic causets from trinet automata and existing quantum gravity theories.

Research on emergence in algorithmic causets is still at an early stage. Progress can be expected from the study of quantitative properties such as dimension and curvature, and from comparisons with stochastic causets. However, we believe that even more exciting results could come from the investigation, by simulation, of qualitative properties. With our work, we hope we have provided some additional arguments in support of the conceptual equation ‘complexity in nature = emergence in computation’. If cellular automata can implement phenomena such as particle interaction and self-replication, is it too ambitious to expect algorithmic causets to implement the mechanisms that eventually trigger and run the biosphere?

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