On a Uniform Framework for the Definition of Stochastic Process Languages* —*Full Version*—

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Abstract. In this paper we present how *Rate Transition Systems* (RTSs) can be used as a unifying framework for the definition of the semantics of stochastic process algebras. RTSs facilitate the *compositional* definition of such semantics exploiting operators on the next state functions which are the functional counterpart of classical process algebra operators. We apply this framework to representative fragments of major stochastic process calculi including $TIPP$, $EMPA$, $PEPA$ and IML and show how they solve the issue of transition multiplicity in a simple and elegant way. We, moreover, show how RTSs help describing different languages, their differences and their similarities. For each calculus, we also show the formal correspondence between the RTSs semantics and the standard SOS one.

1 Introduction

Several stochastic, and in particular Markovian, process algebras have been proposed in the recent past. An overview can be found in [15]. Examples include $TIPP$ [12, 16], $PEPA$ [18], $EMPA$ [3], stochastic π-calculus [23] and, more recently, calculi for Mobile and Service Oriented Computing [10, 6, 7, 22, 4, 8]. The main aim has been the integration of qualitative, behavioural, descriptions with (some) non-functional, e.g. performance, ones in a single mathematical framework, namely that of process algebras. This brought to the combination of two very successful approaches to concurrent systems modelling and analysis, namely Labeled Transition Systems (LTSs), widely used in the framework of process algebra, and Continuous Time Markov Chains (CTMCs), one of the most successful approaches to modelling and analysing system performance. The common feature of the most prominent stochastic process algebra proposals, including all the above mentioned ones, is that the actions used to label transitions are enriched with rates of exponentially distributed random variables (r.v.) characterising their duration[1]. Although the same class of r.v. is assumed in these

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[1] In the literature, some authors assume actions have zero duration, in which case the associated r.v. is interpreted as a *delay*, before the action takes place.

languages, the underlying models and notions are significantly different, in particular w.r.t. the issue of the correct representation of the *race condition* principle for the choice operator, inherited from the theory of CTMCs. This principle implies that an expression like $(\alpha, \lambda).P + (\alpha, \lambda).P$, where there are two different ways of executing α, both with (exponentially distributed duration with) rate λ should model the same behavior as $(\alpha, 2 \cdot \lambda).P$, and not as $(\alpha, \lambda).P$, as it would be the case if one would look at the term as a standard process algebra one. Several, significantly different, approaches have been proposed for addressing the issue of *transition multiplicity* raised by the race condition principle ranging, e.g. from *multi relations* [18], to *proved transition systems* [23, 12], to LTS with *numbered transitions* [15], to *unique rate names* [10, 6].

A different approach has been taken in [14] for IML, a language for Interactive Markov Chains, IMCs, where actions are de-coupled from rates and, consequently, interaction transitions, labelled with actions, are kept separated from Markovian ones, labelled by rates; multi-relations are used for Markovian transitions.

It should also be noticed that some of the most successful approaches, e.g. [18, 14] suffer from technical imprecision in that they define the relevant transition multi-relation as the *least* multi-relation satisfying a set of Structured Operational Semantics (SOS) axioms and rules. Unfortunately, such a least multi-relation turns out to be again a relation, thus failing to formally representing transition multiplicity.

In [19] a variant of LTSs, namely *Rated Transition Systems* (RdTS) has been proposed as a model for the definition of the semantics of Markovian process calculi by relying on the general framework of SGSOS. Moreover, in [19] conditions are put forward for guaranteeing associativity of the parallel composition operator in the SGSOS framework. It is then proved that one cannot guarantee associativity of parallel composition operator up to stochastic bisimilarity when the synchronisation paradigm of CCS is used in combination with the synchronisation rate computation based on *apparent rates* [18]. This implies for instance that parallel composition of Stochastic π is not associative.

In the present paper, we use *Rate Transition Systems* (RTS) a variant of RdTS where the transition relation $\rightarrowtail$ associates to a given process P and a given transition label α a function, say $\mathscr{P}$, mapping each term into a non-negative real number. The transition $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ has the following meaning: if $\mathscr{P}(Q) = v,\ (with\ v \neq 0)$, then Q is reachable from P by executing α, the duration of such execution being exponentially distributed with rate v; if $\mathscr{P}(Q) = 0$, then Q is not reachable from P via α. The approach is somewhat reminiscent of that of Deng et al. [11] where probabilistic process algebra terms are associated to a discrete probability distribution over such terms. RTSs are similar to Continuous Time Markov Decision Processes (CTMDPs) as defined, e.g., in [17, 2] or Continuous Time Probabilistic Automata (CPA) (see [20, 21, 5]) as we shall discuss more in detail later on in this paper. The peculiarity of our approach w.r.t. the above mentioned ones is, on the one hand, compositionality, which, as in [19], is a direct consequence of a structured approach to semantics definition, and, on the other hand, the effective, compositional, exploitation of operators on next state functions $\mathscr{P}$. A pleasant side-effect of this approach is a simple and elegant solution to the transition multiplicity problem. Furthermore, RTSs make it relatively easy to define *associative* parallel composition operators for calculi based on the CCS interaction paradigm. Finally, the possibility of defining different stochastic process languages within a single, uniform framework facilitates reasoning about them, their similarities as well as their major differences.

In this paper we will consider only a limited number of stochastic process calculi, due to space limitations. Moreover, we will focus only on the fragment of each calculus which is relevant for the stochastic extension, leaving out all those operators which are not directly affected by the extension. Finally, we point out that, in the present paper we do not deal with behavioral relations and we focus only on a technique for language definition: this explains why in the title we mention process *languages* and not *calculi*. The reader interested in process equivalences is referred to [8, 9] for some initial results.

The present paper is organized as follows: in Sect. 2 some preliminary notions and definitions are recalled. Sect. 3 introduces the RTS semantics for a simple language for CTMCs. Sect. 4 shows the RTS semantics of significant fragments of major Markovian Process Calculi. Emphasis is put on calculi based on the CSP interaction paradigm, like $TIPP$, $EMPA$ and $PEPA$. A brief discussion of other calculi, based on the CSP interaction paradigm is also provided. The RTS semantics of a language based on the Interactive Markov Chain principle of separating actions from rates is presented in Sect.5. The original SOS definition of the relevant calculi as well as detailed proofs are reported in the appendices.

2 Preliminaries

In the sequel, we let $\mathbb{N}_{\geq 0}$ ($\mathbb{R}_{\geq 0}$, respectively) denote the set $\{n \in \mathbb{N} \mid n \geq 0\}$ ($\{x \in \mathbb{R} \mid x \geq 0\}$, respectively) and, similarly, $\mathbb{N}_{>0}$ ($\mathbb{R}_{>0}$, respectively) denote the set $\{n \in \mathbb{N} \mid n > 0\}$ ($\{x \in \mathbb{R} \mid x > 0\}$, respectively). For set S we let 2^S denote the power-set of S and 2^S_{fin} the set of *finite* subsets of S. In function definitions as well as application *Currying* will be used whenever convenient.

Definition 1 (Negative Exponential Distributions). *A random variable X has a* negative exponential distribution *with* rate λ *if and only if $\mathbb{P}\{X \leq t\} = 1 - e^{\lambda \cdot t}$ for $t > 0$ and 0 otherwise.* •

The expected value of an exponentially distributed r.v. with rate λ is λ^{-1} while its variance is λ^{-2}. The *min* of exponentially distributed independent r.v. $X_1, \ldots, X_n$ with rates $\lambda_1, \ldots, \lambda_n$ respectively is an exponentially distributed r.v. with rate $\lambda_1 + \ldots + \lambda_n$ while the probability that X_j is the *min* is $\frac{\lambda_j}{\lambda_1 + \ldots + \lambda_n}$. The *max* of exponentially distributed r.v. is not exponentially distributed. For the purpose of the present paper, CTMCs are defined as follows:

Definition 2 (Continuous Time Markov Chains). *A* Continuous Time Markov Chain (CTMC) *is a tuple $(S, \mathbf{R})$ where S is a countable non-empty* *set of* states*, and* $\mathbf{R} : S \to S \to I\!R_{\geq 0}$ *is the* rate matrix*, where for all* $s \in S$ *there exists* $K_s < \infty$ *such that* $\sum_{s' \in S} \mathbf{R}\, s\, s' = K_s$. •

We will often use the matrix notation $\mathbf{R}[s, s']$ for $\mathbf{R}\, s\, s'$. $\mathbf{R}[s, s'] > 0$ means that a transition from s to s' can be taken. The sojourn time at state s before taking a transition is an exponentially distributed r.v. with rate $\sum_{s' \in S} \mathbf{R}[s, s']$ and the probability that the transition from s to s' is taken is $\frac{\mathbf{R}[s,s']}{\sum_{s'' \in S} \mathbf{R}[s,s'']}$.

Notice that the above definition allows $\mathbf{R}[s, s] > 0$, i.e. self-loops are allowed, which is not the case in traditional definitions of CTMCs. The following proposition, proved in Sect. A, shows that, as long as traditional measures of CTMCs like transient (and consequently steady state) probabilities are concerned, this more liberal definition does not affect the meaning of the CTMC and, in fact, self-loops can be removed (i.e. $\mathbf{R}[s, s]$ set to zero) or added without affecting transient and steady state probability analysis results.

Proposition 1. *The transient behaviour of CTMC* $C = (S, \mathbf{R})$ *with* $\mathbf{R}[\bar{s}, \bar{s}] > 0$ *for some* $\bar{s} \in S$ *coincides with that of CTMC* $\tilde{C} = (S, \tilde{\mathbf{R}})$*, such that:*

$$\tilde{\mathbf{R}}[s, s'] \;=_{\mathrm{def}}\; \begin{cases} 0 & \textit{if } s = s' \\ \mathbf{R}[s, s'] & \textit{otherwise} \end{cases}$$

□

As a consequence of the above result, the infinitesimal generator matrix representation of CTMCs, traditionally used for CTMCs without self-loops, can be safely used also for those with such loops.

For countable non-empty set S, we consider the set $S \to \mathbb{R}_{\geq 0}$ of total functions from S to $\mathbb{R}_{\geq 0}$. We let $\mathscr{P}, \mathscr{Q}, \mathscr{R}, \ldots$ range over $S \to \mathbb{R}_{\geq 0}$. We let $[]$ denote the 0 constant function in $S \to \mathbb{R}_{\geq 0}$, i.e. $[]\, s \;=_{\mathrm{def}}\; 0$ for all $s \in S$; moreover given $s_1, \ldots, s_n \in S$ and, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{>0}$ we let $[s_1 \mapsto \lambda_1, \ldots, s_n \mapsto \lambda_n]$ denote the function in $S \to \mathbb{R}_{\geq 0}$ which maps s_1 to $\lambda_1, \ldots, s_n$ to λ_n and any $s \in S \setminus \{s_1, \ldots, s_n\}$ to 0. The following definition characterises *Rate Transition Systems* [8, 9].

Definition 3 (Rate Transition Systems). *A* Rate Transition System *(RTS) is a tuple* $(S, A, \rightarrowtail)$ *where* S *is a countable non-empty set of* states*,* A *is a countable non-empty set of* labels *and* $\rightarrowtail \subseteq S \times A \times (S \to I\!R_{\geq 0})$ *is the* transition relation*.*

In the sequel RTSs will be denoted by $\mathcal{R}$, $\mathcal{R}_1$, $\mathcal{R}'$,... As usual, we let $s \overset{\alpha}{\rightarrowtail} \mathscr{P}$ denote $(s, \alpha, \mathscr{P}) \in \rightarrowtail$. Intuitively, $s_1 \overset{\alpha}{\rightarrowtail} \mathscr{P}$ and $\mathscr{P}(s_2) = \lambda \neq 0$ means that s_2 is reachable from s_1 via the execution of α; moreover, the duration of such an execution is characterised by a random variable whose distribution function is negative exponential with rate λ. On the other hand, $\mathscr{P}(s_2) = 0$ means that s_2 is not reachable from s_1 via α.

Definition 4 (Σ_S). Σ_S *denotes the subset of* $S \to I\!R_{\geq 0}$ *including only all functions expressed using the* $[\ldots]$ *notation, i.e.* $\mathscr{P} \in \Sigma_S$ *if and only if* $\mathscr{P} = []$ *or there exist* $n \in I\!N_{>0}$, $s_1, \ldots, s_n \in S$ *and* $\lambda_1, \ldots, \lambda_n \in I\!R_{>0}$ *such that* $\mathscr{P} = [s_1 \mapsto \lambda_1, \ldots, s_n \mapsto \lambda_n]$ •

We equip Σ_S with a few useful operations, i.e. $+ : \Sigma_S \times \Sigma_S \to S \to \mathbb{R}_{\geq 0}$ with $(\mathscr{P} + \mathscr{Q})\, s \ =_{\mathrm{def}} \ (\mathscr{P}\, s) + (\mathscr{Q}\, s)$ and $\bigoplus : \Sigma_S \to 2^S \to \mathbb{R}_{\geq 0}$ with $\bigoplus \mathscr{P}\, C \ =_{\mathrm{def}} \sum_{s \in C}(\mathscr{P} s)$ and we use the shorthand $\oplus\mathscr{P}$ for $\bigoplus \mathscr{P}\, S$. The proposition below trivially follows from the relevant definitions:

Proposition 2. *(i) All functions in Σ_S yield zero almost everywhere, i.e. for all $\mathscr{P} \in \Sigma_S$ the set $\{s \in S \mid \mathscr{P}\, s \neq 0\}$ is finite; (ii) Σ_S is* closed *under $+$, i.e. $+ : \Sigma_S \to \Sigma_S \to \Sigma_S$.* □

Notice that $\bigoplus$ is well defined due to Proposition 2(i).

Definition 5. *Let $\mathcal{R} = (S, A, \rightarrowtail)$ be an RTS, then:*

i) $\mathcal{R}$ is total *if for all $s \in S$ and $\alpha \in A$ there exists $\mathscr{P} \in (S \to \mathbb{R}_{\geq 0})$ such that $s \overset{\alpha}{\rightarrowtail} \mathscr{P}$;*
ii) $\mathcal{R}$ is functional[2] *if for all $s \in S$, $\alpha \in A$, and $\mathscr{P}, \mathscr{Q} \in (S \to \mathbb{R}_{\geq 0})$ we have: $s \overset{\alpha}{\rightarrowtail} \mathscr{P}, s \overset{\alpha}{\rightarrowtail} \mathscr{Q} \Longrightarrow \mathscr{P} = \mathscr{Q}$;*
iii) $\mathcal{R}$ is well formed *if $\rightarrowtail \subseteq S \times A \times \Sigma_S$.* •

Discussion

It is worth noting that RTSs are a slight generalization of Continuous Time Markov Decision Processes (CTMDPs) as defined by Hermanns and Johr [17] and Continuous Time Probabilistic Automata, as defined in [21]. In [17, 21], in fact, the transition relation is a subset of $S \times A \times (S \to \mathbb{R}_{\geq 0})$, i.e. it is *not* required to be a *function* in $S \times A \to (S \to \mathbb{R}_{\geq 0})$, but sets S and A are required to be *finite* and in [17] an *initial state* is assumed as well. There is also a direct relationship between RTSs and Continuous Time Probabilistic Automata proposed in [20], although the latter are studied in a language theoretic framework: the element $a_{i,j}(x)$ of the infinitesimal matrix used in [20] coincides, in our approach, with $(\mathscr{P}\, j)$ for $i \overset{x}{\rightarrowtail} \mathscr{P}$. Finally, the Continuous Time Probabilistic Automata used in [5] are based on standard automata, where transitions are elements of $S \times S$ and have a rate and no label associated. In [19] *Rated Transition Systems* (RdTSs) are proposed. They coincide with the class of *functional* RTS: the transition relation is required to be a *function* in $S \times A \times S \to \mathbb{R}_{\geq 0} = S \times A \to (S \to \mathbb{R}_{\geq 0})$. In [2] CTMDPs are defined as tuples $(S, A, \rightarrowtail)$ where S and A are *finite* sets and $\rightarrowtail$ is a function in $S \times A \times S \to \mathbb{R}_{\geq 0}$, so they are a specialisation of RdTS.

We point out that, as we shall see, RTSs can be used to model (passive) action *weights*, e.g. in $EMPA$ or $PEPA$ as well as *interactive* transitions of Interactive Markov Chains in a natural way.

Finally, we recall that a specific equivalence relation has been defined on RTSs, namely *Rate Aware Bisimilarity*, using the notion of *Rate Aware Bisimulation* relation—the RTS counterpart of standard process Bisimulation relation. We refer the reader to [9, 8] where also the relationship between Rate

[2] *Fully-stochastic* according to the terminology used in [9].

Aware Bisimilarity and Strong Markovian Bisimilarity is investigated.

In the rest of the present paper we will consider only *well-formed* RTSs, since they are powerful enough as semantic model for the stochastic process calculi we deal with.

Definition 6 (Derivatives). *Let $\mathcal{R} = (S, A, \rightarrowtail)$ be an RTS; for sets $S' \subseteq S$ and $A' \subseteq A$, the set of derivatives of S'* through *A', denoted $Der(S', A')$, is the smallest set such that:*

- $S' \subseteq Der(S', A')$,
- *if $s \in Der(S', A')$ and there exists $\alpha \in A'$ and $\mathscr{Q} \in \Sigma_S$ such that $s \overset{\alpha}{\rightarrowtail} \mathscr{Q}$ then $\{s' \in S \mid \mathscr{Q}(s') \neq 0\} \subseteq Der(S', A')$.* •

Definition 7 (Derived CTMC). *Let $\mathcal{R} = (S, A, \rightarrowtail)$ be a* functional *RTS; for $S' \subseteq S$, the CTMC of S', when one considers only labels in* finite *set $A' \subseteq A$ is defined as $CTMC[S', A'] =_{\text{def}} (Der(S', A'), \mathbf{R})$ where for all $s_1, s_2 \in Der(S', A')$:*

$$\mathbf{R}[s_1, s_2] =_{\text{def}} \sum_{\substack{\alpha \in A' \\ s_1 \overset{\alpha}{\rightarrowtail} \mathscr{P}}} \mathscr{P}(s_2).$$

We write $Der(s, A')$ and $CTMC[s, A']$ when $S' = \{s\}$.

The semantics of stochastic process calculi are often defined in the literature by means of Structured Operational Semantics (SOS) which characterize transition systems or *multi*-transition systems, i.e. transition systems where the transition relation is instead a *multi*-relation. Such (multi-)transitions are usually labelled by *rates* $\lambda \in \mathbb{R}_{>0}$, and in some approaches they are also labelled with *action* labels drawn from some set A, while in some other cases they can be labelled by action labels and *weights* $*_w$, with $w \in \mathbb{R}_{>0}$. In such LTSs there can be two or more transitions with (the same action label and) different rates (or weights) from a state s_1 to a state s_2; in case of multi-transition systems such distinct transitions may even have the same rate (or weight). Henceforth we let $\mathbf{rt}(s_1, s_2)$ ($\mathbf{rt}_a(s_1, s_2)$, $\mathbf{wt}_a(s_1, s_2)$, respectively) denote the *cumulative* rate over *all* transitions (rate, weight over all a labelled transitions, respectively) from s_1 to s_2, as defined below, using $\{\!|\ \_\ |\!\}$ as a notation for multi-sets, and $\xrightarrow{\lambda}$ ($\xrightarrow{a,\lambda}$, $\xrightarrow{a,*_w}$ respectively) for a generic transition (a-labelled transition):

Definition 8 (Cumulative rate/weight).

$$\mathbf{rt}(s_1, s_2) =_{\text{def}} \sum \{\!|\ \lambda | s_1 \xrightarrow{\lambda} s_2 |\!\}$$

$$\mathbf{rt}_a(s_1, s_2) =_{\text{def}} \sum \{\!|\ \lambda | s_1 \xrightarrow{a,\lambda} s_2 |\!\}$$

$$\mathbf{wt}_a(s_1, s_2) =_{\text{def}} \sum \{\!|\ w | s_1 \xrightarrow{a,*_w} s_2 |\!\}$$

with $\sum \{\!|\,|\!\} =_{\text{def}} 0$ •

3 A Language for CTMCs

In this section we define a simple language for CTMCs, in a similar way as in [15].

3.1 Syntax

The set $\mathcal{P}_{CTMC}$ of CTMC terms is defined by the following grammar:

$$P ::= \mathbf{nil} \;\Big|\; \lambda.P \;\Big|\; P+P \;\Big|\; X$$

where $\lambda \in \mathbb{R}_{>0}$ and X is a constant defined by means of an equation of the form $X \stackrel{\Delta}{=} P$ where constants $X, X_1, X', \ldots$ may occur only guarded in P, i.e. under the scope of a prefix $\lambda.\_$.

3.2 Semantics

In order to give an RTS semantics to the calculus we first of all choose the set $\mathcal{A}_{CTMC} =_{\mathrm{def}} \{\surd\}$ as labels set; transitions have no action labels in standard CTMCs. The transition relation $\rightarrowtail$, is defined in Fig. 1.

$$\frac{}{\mathbf{nil} \stackrel{\surd}{\rightarrowtail} []} \quad \frac{}{\lambda.P \stackrel{\surd}{\rightarrowtail} [P \mapsto \lambda]} \quad \frac{P \stackrel{\surd}{\rightarrowtail} \mathscr{P}, Q \stackrel{\surd}{\rightarrowtail} \mathscr{Q}}{P+Q \stackrel{\surd}{\rightarrowtail} \mathscr{P}+\mathscr{Q}} \quad \frac{P \stackrel{\surd}{\rightarrowtail} \mathscr{P}, X \stackrel{\Delta}{=} P}{X \stackrel{\surd}{\rightarrowtail} \mathscr{P}}$$

Fig. 1. Rules for the CTMC Language.

Intuitively, from Fig. 1 it is clear that there is no transition from **nil** to any other state, while there is a single transition from $\lambda.P$ to P and λ is the rate associated to such a transition. The rule for choice postulates that if there is a transition from P to a state, say R, with rate $(\mathscr{P}R)$ and a transition from Q to the same state R, with rate $(\mathscr{Q}R)$, then there is a transition from $P+Q$ to R with rate $(\mathscr{P}R) + (\mathscr{Q}R)$.

Remark
Notice that, for term $P+Q$, if there is a transition *only* from P to R (i.e. not from Q to R) then $(\mathscr{Q}R) = 0$. Similarly, $(\mathscr{P}R) = 0$ if there is only a transition from Q to R. If, instead, there is *both* a transition from P to R (i.e. $(\mathscr{P}R) > 0$) *and* a transition from Q to R (i.e. $(\mathscr{Q}R) > 0$), then the cumulative rate $(\mathscr{P}R) + (\mathscr{Q}R)$ will be associated directly to the transition from $P+Q$ to R. The use of RTSs, in particular in the rule for choice, incorporates the *race condition* principle and solves the related *transition multiplicity* issue in a simple and elegant way. In fact, from Fig. 1, for $R_1 \neq R_2$ we get $\lambda.R_1 + \mu.R_2 \stackrel{\surd}{\rightarrowtail} [R_1 \mapsto \lambda, R_2 \mapsto \mu]$

where $(\oplus[R_1 \mapsto \lambda, R_2 \mapsto \mu]) = \lambda + \mu$ is the exit rate of state $\lambda.R_1 + \mu.R_2$ while $\lambda/(\lambda+\mu)$ and $\mu/(\lambda+\mu)$ are the probabilities of moving to R_1 and R_2, respectively. If $R_1 = R_2 = R$ then we get $\lambda.R + \mu.R \stackrel{\surd}{\rightarrowtail} [R \mapsto \lambda + \mu]$ and if, moreover, $\lambda = \mu$, we get $\lambda.R + \lambda.R \rightarrowtail [R \mapsto 2\lambda]$.

The following proposition can be easily proven by derivation induction using Proposition 2(ii):

Proposition 3. *For all $P \in \mathcal{P}_{CTMC}$ and $\mathscr{P} \in \mathcal{P}_{CTMC} \to I\!R_{\geq 0}$, if $P \rightarrowtail \mathscr{P}$ can be derived from the rules of Fig. 1, then $\mathscr{P} \in \Sigma_{\mathcal{P}_{CTMC}}$.* □

Definition 9 (Formal semantics of the Language for CTMCs). *The formal semantics of the calculus for CTMCs is the RTS $\mathcal{R}_{CTMC} =_{\text{def}} (\mathcal{P}_{CTMC}, \mathcal{A}_{CTMC}, \rightarrowtail)$ where $\rightarrowtail \subseteq \mathcal{P}_{CTMC} \times \mathcal{A}_{CTMC} \times \Sigma_{\mathcal{P}_{CTMC}}$ is the least relation satisfying the rules of Fig. 1.* •

The following theorem characterises the structure of $\mathcal{R}_{CTMC}$

Theorem 1. *$\mathcal{R}_{CTMC}$ is total and functional.* □

The CTMC associated to a given term $P \in \mathcal{P}_{CTMC}$, $CTMC[P, \{\surd\}]$ is generated according to Def. 7. As a corollary of Theorem 1 we get that whenever $P \stackrel{\surd}{\rightarrowtail} \mathscr{P}$, the exit rate of P is given by $\oplus\mathscr{P}$ and $\mathscr{P}$ is the row of the rate matrix corresponding to P.

4 Fully Markovian Stochastic Process Calculi

We first introduce some additional notation. Let S and A be countable non-empty sets. We define function $\chi : S \to S \to I\!R_{\geq 0}$ as $\chi\, s =_{\text{def}} [s \mapsto 1]$. Let, moreover, $\_ \otimes_{\_} \_ : 2^A_{fin} \to S \to S \to S$ be a total function and let us define, with a little bit of overloading, function $\_ \otimes_{\_} \_ : 2^A_{fin} \to (S \to I\!R_{\geq 0}) \to (S \to I\!R_{\geq 0}) \to (S \to I\!R_{\geq 0})$ as follows:

$$(\mathscr{P} \otimes_L \mathscr{Q})\, s =_{\text{def}} \begin{cases} (\mathscr{P}\, s_1) \cdot (\mathscr{Q}\, s_2), & \text{if } \exists s_1, s_2 \in S.\ s = s_1 \otimes_L s_2 \\ 0 & \text{, otherwise} \end{cases}$$

We also define function $\_ \cdot \_ / \_ : (S \to I\!R_{\geq 0}) \to I\!R_{\geq 0} \to I\!R_{\geq 0} \to S \to I\!R_{\geq 0}$ as follows:

$$(\mathscr{P} \cdot \frac{x}{y})\, s =_{\text{def}} \begin{cases} (\mathscr{P}\, s) \cdot \frac{x}{y}, & \text{if } y \neq 0 \\ 0 & \text{, otherwise} \end{cases}$$

The proposition below trivially follows from the relevant definitions:

Proposition 4. *Σ_S is* closed *under the operations $\chi, (\_ \otimes_{\_} \_)$, and $\_ \cdot \_/\_$, i.e. $\chi : S \to \Sigma_S$, $(\_ \otimes_{\_} \_) : 2^A_{fin} \to \Sigma_S \to \Sigma_S \to \Sigma_S$, and $\_ \cdot \_/\_ : \Sigma_S \to I\!R_{\geq 0} \to I\!R_{\geq 0} \to \Sigma_S$.* □

4.1 $TIPP_k$

Here we consider a kernel language $TIPP_k$ of the version of $TIPP$ presented in [16][3]. In Sect. B.2 we recall the SOS of $TIPP$, as defined in [16], for the fragment $TIPP_k$ we focus on.

Syntax The set $\mathcal{P}_{TIPP_k}$ of $TIPP$ terms we consider is defined by the following grammar:

$$P ::= \mathbf{stop} \;\Big|\; (a,\lambda).P \;\Big|\; P[]P \;\Big|\; P|[L]|P \;\Big|\; X$$

where $a \in \mathcal{A}_{TIPP_k} \cup \{\mathtt{i}\}$, with $\mathcal{A}_{TIPP_k}$ a countable set of *actions* and $\mathtt{i}$ the internal action, $\lambda \in \mathbb{R}_{>0}$, finite *synchronisation* set $L \in 2_{fin}^{\mathcal{A}_{TIPP_k}}$, and X is a constant defined by means of an equation of the form $X \overset{\Delta}{=} P$ where constants $X, X_1, X', \ldots$ may only occur guarded in P, i.e. under the scope of a prefix $(a,\lambda).\_$.

$$\frac{}{\mathbf{stop} \overset{a}{\rightarrowtail} []} \quad \frac{}{(a,\lambda).P \overset{a}{\rightarrowtail} [P\mapsto\lambda]} \quad \frac{a\neq b}{(a,\lambda).P \overset{b}{\rightarrowtail} []}$$

$$\frac{P \overset{a}{\rightarrowtail} \mathscr{P}, Q \overset{a}{\rightarrowtail} \mathscr{Q}}{P\,[]\,Q \overset{a}{\rightarrowtail} \mathscr{P}+\mathscr{Q}} \quad \frac{P \overset{a}{\rightarrowtail} \mathscr{P}, X \overset{\Delta}{=} P}{X \overset{a}{\rightarrowtail} \mathscr{P}}$$

$$\frac{P \overset{a}{\rightarrowtail} \mathscr{P}, Q \overset{a}{\rightarrowtail} \mathscr{Q}, a\notin L}{P|[L]|Q \overset{a}{\rightarrowtail} (\mathscr{P}|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q})} \quad \frac{P \overset{a}{\rightarrowtail} \mathscr{P}, Q \overset{a}{\rightarrowtail} \mathscr{Q}, a\in L}{P|[L]|Q \overset{a}{\rightarrowtail} \mathscr{P}|[L]|\mathscr{Q}}$$

Fig. 2. Rules for $TIPP_k$.

Semantics The transition relation $\rightarrowtail$ for $TIPP_k$ is defined in Fig. 2. In the rules, the generic functions χ and $\otimes$ on S are instantiated with specific functions for $\mathcal{P}_{TIPP_k}$. In particular the specific function $|[\,]|$ is used in place of the generic function $\otimes$; the specific function $\_|[\_]|\_ : 2_{fin}^{\mathcal{A}_{TIPP_k}} \to \mathcal{P}_{TIPP_k} \to \mathcal{P}_{TIPP_k} \to \mathcal{P}_{TIPP_k}$ is just the syntactical constructor for parallel composition on $TIPP$ terms. The rule for interleaving ensures that all interesting continuations of $P\,|[L]|\,Q$ are of the form $R\,|[L]|\,Q$ where $P \overset{a}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\,R) > 0$, for some

[3] In [16] the synchronisation rate is defined as the product of those of the synchronising actions, as opposed to the original definition of $TIPP$, given in [12], where, instead, such rate is the *max* of the component rates

$\mathscr{P}$ and $a \notin L$, or of the form $P\,|[L]|\,R$ where $Q \stackrel{a}{\rightarrowtail} \mathscr{Q}$ and $(\mathscr{Q}\,R) > 0$, for some $\mathscr{Q}$ and $a \notin L$. The rule for synchronisation, instead, implements the *rate multiplication* principle of $TIPP$: if $a \in L$, $P \stackrel{a}{\rightarrowtail} \mathscr{P}$, $Q \stackrel{a}{\rightarrowtail} \mathscr{Q}$, $(\mathscr{P}\,R_P) = \lambda_P > 0$, and $(\mathscr{Q}\,R_Q) = \lambda_Q > 0$, then $P\,|[L]|\,Q$ evolves, via a, to $R_P\,|[L]|\,R_Q$ with rate $\lambda_P \cdot \lambda_Q$.

The following proposition can be easily proven by derivation induction using Proposition 2(ii) and Proposition 4:

Proposition 5. *For all $P \in \mathcal{P}_{TIPP_k}$, $a \in \mathcal{A}_{TIPP_k} \cup \{\mathtt{i}\}$ and $\mathscr{P} \in \mathcal{P}_{TIPP_k} \to I\!R_{\geq 0}$, if $P \stackrel{a}{\rightarrowtail} \mathscr{P}$ can be derived from the rules of Fig. 2, then $\mathscr{P} \in \Sigma_{\mathcal{P}_{TIPP_k}}$.* □

Definition 10 (Formal semantics of $TIPP_k$). *The formal semantics of $TIPP_k$ is the RTS $\mathcal{R}_{TIPP_k} =_{\mathrm{def}} (\mathcal{P}_{TIPP_k}, \mathcal{A}_{TIPP_k} \cup \{\mathtt{i}\}, \rightarrowtail)$ where $\rightarrowtail \subseteq \mathcal{P}_{TIPP_k} \times (\mathcal{A}_{TIPP_k} \cup \{\mathtt{i}\}) \times \Sigma_{\mathcal{P}_{\mathrm{TIPP_k}}}$ is the least relation satisfying the rules of Fig. 2.* •

The following theorem characterises the structure of $\mathcal{R}_{TIPP_k}$.

Theorem 2. *$\mathcal{R}_{TIPP_k}$ is total and functional.*

Corollary 1. *For all $P \in \mathcal{P}_{TIPP_k}, a \in \mathcal{A}_{TIPP_k} \cup \{\mathtt{i}\}$ there exists a* unique *$\mathscr{P}$ such that $P \stackrel{a}{\rightarrowtail} \mathscr{P}$.*

The following theorem establishes the formal correspondence between the RTS semantics of $TIPP_k$ and the semantics definition given in [16].

Theorem 3. *For all $P, Q \in \mathcal{P}_{TIPP_k}$, $a \in \mathcal{A}_{TIPP_k} \cup \{\mathtt{i}\}$, and unique $\mathscr{P} \in \Sigma_{\mathcal{P}_{TIPP_k}}$ such that $P \stackrel{a}{\rightarrowtail} \mathscr{P}$ the following holds: $(\mathscr{P}\,Q) = \mathbf{rt}_a(P, Q)$* □

4.2 $EMPA_k$

In this section we consider a kernel language, $EMPA_k$, of $EMPA$ [1] which shows the key principle of $EMPA$ Markovian synchronization. In each interaction, it is required that there is a single active action while all other actions taking part in the interaction are required to be passive. Consequently, the rate of the interaction is that of the unique active action. Passive actions are equipped with weights. We consider only the features of the exponentially timed kernel of $EMPA$. In particular, here we do not address features of $EMPA$ such as prioritized, immediate actions and probabilities. In Sect. B.3 we recall the SOS of $EMPA$, as defined in [1], for the fragment $EMPA_k$ we focus on.

Syntax The set $\mathcal{P}_{EMPA_k}$ of $EMPA$ terms we consider is defined by the following grammar:

$$P ::= \underline{\mathbf{0}} \;\Big|\; <a, \lambda>.P \;\Big|\; <a, *_w>.P \;\Big|\; P + P \;\Big|\; P||_L P \;\Big|\; X$$

where $a \in \mathcal{A}_{EMPA_k} \cup \{\tau\}$ with τ the internal action, with $\mathcal{A}_{EMPA_k}$ countable set of *actions*, $\lambda, w \in \mathbb{R}_{>0}$, finite *synchronisation* set $L \in 2^{\mathcal{A}_{EMPA_k}}_{fin}$, and X is a constant defined by means of an equation of the form $X \stackrel{\Delta}{=} P$ where constants $X, X_1, X', \ldots$ may occur only guarded in P, i.e. under the scope of a prefix $< a, \lambda > ._$ or $< a, *_w > ._$.

$$\frac{}{\underline{\mathbf{0}} \stackrel{\alpha}{\rightarrowtail} []} \quad \frac{a \neq \alpha}{<a,\lambda>.P \stackrel{\alpha}{\rightarrowtail} []} \quad \frac{a* \neq \alpha}{<a,*_w>.P \stackrel{\alpha}{\rightarrowtail} []}$$

$$\frac{}{<a,\lambda>.P \stackrel{a}{\rightarrowtail} [P \mapsto \lambda]} \quad \frac{}{<a,*_w>.P \stackrel{a*}{\rightarrowtail} [P \mapsto w]}$$

$$\frac{P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}, Q \stackrel{\alpha}{\rightarrowtail} \mathscr{Q}}{P + Q \stackrel{\alpha}{\rightarrowtail} \mathscr{P} + \mathscr{Q}} \quad \frac{P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}, X \stackrel{\Delta}{=} P}{X \stackrel{\alpha}{\rightarrowtail} \mathscr{P}}$$

$$\frac{P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}, Q \stackrel{\alpha}{\rightarrowtail} \mathscr{Q}, (n\,\alpha) \notin L}{P ||_L Q \stackrel{\alpha}{\rightarrowtail} (\mathscr{P} ||_L (\chi\, Q)) + ((\chi\, P) ||_L \mathscr{Q})} \quad \frac{P \stackrel{a*}{\rightarrowtail} \mathscr{P}, Q \stackrel{a*}{\rightarrowtail} \mathscr{Q}, a \in L}{P ||_L Q \stackrel{a*}{\rightarrowtail} \mathscr{P} ||_L \mathscr{Q} \cdot \frac{(\oplus\,\mathscr{P}) + (\oplus\,\mathscr{Q})}{(\oplus\,\mathscr{P}) \cdot (\oplus\,\mathscr{Q})}}$$

$$\frac{P \stackrel{a}{\rightarrowtail} \mathscr{P}_o, P \stackrel{a*}{\rightarrowtail} \mathscr{P}_i, Q \stackrel{a*}{\rightarrowtail} \mathscr{Q}_i, Q \stackrel{a}{\rightarrowtail} \mathscr{Q}_o, a \in L}{P ||_L Q \stackrel{a}{\rightarrowtail} \mathscr{P}_o ||_L \mathscr{Q}_i \cdot \frac{1}{\oplus\,\mathscr{Q}_i} + \mathscr{P}_i ||_L \mathscr{Q}_o \cdot \frac{1}{\oplus\,\mathscr{P}_i}}$$

Fig. 3. Rules for $EMPA_k$.

Semantics In order to impose the $EMPA$ rules of interaction, i.e. forbidding synchronization between active actions, it is convenient to extend the set $\mathcal{A}_{EMPA_k}$ as follows: $\mathcal{A}_{EMPA_{k*}} =_{\text{def}} \mathcal{A}_{EMPA_k} \cup \{\tau\} \cup \{a* \mid a \in \mathcal{A}_{EMPA_k}\}$. We let $\alpha, \alpha_1, \alpha', \ldots$ range over $\mathcal{A}_{EMPA_{k*}}$ and we define $n : \mathcal{A}_{EMPA_{k*}} \to \mathcal{A}_{EMPA_k}$ with $n\,a* = n\,a = a$

The transition relation $\rightarrowtail$, is defined in Fig. 3. In the rules, the generic functions χ and $\otimes$ on S are instantiated with specific functions on $\mathcal{P}_{EMPA_k}$. In particular the specific function $||$ is used in place of the generic function $\otimes$; the specific function $\_||\_\_ : 2^{\mathcal{A}_{EMPA_k}}_{fin} \to \mathcal{P}_{EMPA_k} \to \mathcal{P}_{EMPA_k} \to \mathcal{P}_{EMPA_k}$ is just the syntactical constructor for synchronisation on $EMPA$ terms. The rule for interleaving ensures that all continuations of $P\,||_L\,Q$ are of the form $R\,||_L\,Q$ where $P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\,R) > 0$, for some α and $\mathscr{P}$, or of the form $P\,||_L\,R$ where $Q \stackrel{\alpha}{\rightarrowtail} \mathscr{Q}$ and $(\mathscr{Q}\,R) > 0$, for some α and $\mathscr{Q}$. Notice that α can also be $a*$ for some a, in which case $\mathscr{P}$ or $\mathscr{Q}$ yield weights.

The first rule for synchronization models the "passive side" of the *asymmetry* principle of $EMPA$: for $a \in L$, if $P \overset{a*}{\rightarrowtail} \mathscr{P}$, $Q \overset{a*}{\rightarrowtail} \mathscr{Q}$, $(\mathscr{P}\, R_P) = w_P > 0$, and $(\mathscr{Q}\, R_Q) = w_Q > 0$, then $P \,||_L\, Q$ evolves to $R_P \,||_L\, R_Q$ with weight $w_P \cdot w_Q \cdot \frac{(\oplus\mathscr{P})+(\oplus\mathscr{Q})}{(\oplus\mathscr{P})\cdot(\oplus\mathscr{Q})}$, under the assumption that $(\oplus\mathscr{P})$ and $(\oplus\mathscr{Q})$ are positive, otherwise $P \,||_L\, Q$ leads to $[]$ via $a*$. Notice that $(\oplus\mathscr{P})$ ($(\oplus\mathscr{Q})$, respectively) is the total weight of a in P (Q, respectively). The second rule for synchronization, instead, implements the *asymmetry* principle of $EMPA$: the transitions concerning the active role of a in P, i.e. $P \overset{a}{\rightarrowtail} \mathscr{P}_o$, are paired with the transitions concerning the "passive role" of a in Q, i.e. $Q \overset{a*}{\rightarrowtail} \mathscr{Q}_i$, and the resulting function $\mathscr{P}_o \,||_L\, \mathscr{Q}_i$ is normalized with the positive weight of a in Q, i.e. $(\oplus\mathscr{Q}_i)$, and vice-versa.

The following proposition can be easily proven by derivation induction using Proposition 2(ii) and Proposition 4:

Proposition 6. *For all* $P \in \mathcal{P}_{EMPA_k}$, $\alpha \in \mathcal{A}_{EMPA_{k*}} \cup \{\tau\}$ *and* $\mathscr{P} \in \mathcal{P}_{EMPA_k} \to I\!R_{\geq 0}$, *if* $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ *can be derived from the rules of Fig. 3, then* $\mathscr{P} \in \Sigma_{\mathcal{P}_{EMPA_k}}$. □

Definition 11 (Formal semantics of $EMPA_k$). *The formal semantics of* $EMPA_k$ *is the RTS* $\mathcal{R}_{EMPA_k} =_{\text{def}} (\mathcal{P}_{EMPA_k}, \mathcal{A}_{EMPA_{k*}} \cup \{\tau\}, \rightarrowtail)$ *where* $\rightarrowtail \subseteq \mathcal{P}_{EMPA_k} \times (\mathcal{A}_{EMPA_{k*}} \cup \{\tau\}) \times \Sigma_{\mathcal{P}_{EMPA_k}}$ *is the least relation satisfying the rules of Fig. 3.* •

Theorem 4. $\mathcal{R}_{EMPA_k}$ *is total and functional.*

Corollary 2. *For all* $P \in \mathcal{P}_{EMPA_k}, \alpha \in \mathcal{A}_{EMPA_{k*}} \cup \{\tau\}$ *there exists a* unique $\mathscr{P}$ *such that* $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$.

Let $weight(P, a)$ be defined as follows:

$$weight(P,a) \;=_{\text{def}}\; \sum \{\!| w \in \mathbb{R}_{>0} | \exists P' \in \mathcal{P}_{EMPA_k}.\, P \xrightarrow{a,*w} P' |\!\}$$

The following theorem establishes the formal correspondence between the RTS semantics of $EMPA_k$ and the semantics definition given in [1].

Theorem 5. *For all* $P, Q \in \mathcal{P}_{EMPA_k}$, $a, a* \in \mathcal{A}_{EMPA_{k*}}$ *or* $a = \tau$, *and unique functions* $\mathscr{P}, \mathscr{P}' \in \Sigma_{\mathcal{P}_{EMPA_k}}$ *such that* $P \overset{a}{\rightarrowtail} \mathscr{P}$ *and* $P \overset{a*}{\rightarrowtail} \mathscr{P}'$ *the following holds:* $(\mathscr{P}\, Q) = \mathbf{rt}_a(P, Q)$, $(\mathscr{P}'\, Q) = \mathbf{wt}_a(P, Q)$, *and* $(\oplus\mathscr{P}') = weight(P, a)$ □

4.3 $PEPA_k$

The RTS semantics of the full $PEPA$ [18] calculus can be found in [9]. Here we confine our presentation to the kernel language $PEPA_k$. In Sect. B.4 we recall the SOS of $PEPA$, as defined in [18], for the fragment $PEPA_k$ we focus on.

Syntax The set $\mathcal{P}_{PEPA_k}$ of $PEPA$ terms we consider is defined by the following grammar:

$$P ::= (\alpha, \lambda).P \;\Big|\; P + P \;\Big|\; P \bowtie_L P \;\Big|\; X$$

where $\alpha \in \mathcal{A}_{PEPA_k}$, with $\mathcal{A}_{PEPA_k}$ a countable set of *action types*, $\lambda \in \mathbb{R}_{>0}$, finite *co-operation* set $L \in 2_{fin}^{\mathcal{A}_{PEPA_k}}$, and X is a constant defined by means of an equation of the form $X \stackrel{\Delta}{=} P$ where constants $X, X_1, X', \ldots$ may occur only guarded in P, i.e. under the scope of a prefix $(\alpha, \lambda).\_$.

$$\frac{}{(\alpha,\lambda).P \stackrel{\alpha}{\rightarrowtail} [P \mapsto \lambda]} \quad \frac{\alpha \neq \beta}{(\alpha,\lambda).P \stackrel{\beta}{\rightarrowtail} []} \quad \frac{P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}, Q \stackrel{\alpha}{\rightarrowtail} \mathscr{Q}}{P+Q \stackrel{\alpha}{\rightarrowtail} \mathscr{P}+\mathscr{Q}} \quad \frac{P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}, X \stackrel{\Delta}{=} P}{X \stackrel{\alpha}{\rightarrowtail} \mathscr{P}}$$

$$\frac{P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}, Q \stackrel{\alpha}{\rightarrowtail} \mathscr{Q}, \alpha \notin L}{P \bowtie_L Q \stackrel{\alpha}{\rightarrowtail} (\mathscr{P} \bowtie_L (\chi\, Q)) + ((\chi\, P) \bowtie_L \mathscr{Q})} \quad \frac{P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}, Q \stackrel{\alpha}{\rightarrowtail} \mathscr{Q}, \alpha \in L}{P \bowtie_L Q \stackrel{\alpha}{\rightarrowtail} \mathscr{P} \bowtie_L \mathscr{Q} \cdot \frac{min\{\oplus\mathscr{P}, \oplus\mathscr{Q}\}}{\oplus\mathscr{P} \cdot \oplus\mathscr{Q}}}$$

Fig. 4. Rules for $PEPA$.

Semantics The transition relation $\rightarrowtail$, is defined in Fig. 4. In the rules, the generic functions χ and $\otimes$ on S are instantiated with specific functions on $\mathcal{P}_{PEPA_k}$. In particular the specific function $\bowtie$ is used in place of the generic function $\otimes$; the specific function $\_\bowtie\_\_ : 2_{fin}^{\mathcal{A}_{PEPA_k}} \to \mathcal{P}_{PEPA_k} \to \mathcal{P}_{PEPA_k} \to \mathcal{P}_{PEPA_k}$ is just the syntactical constructor for co-operation on $PEPA$ terms. The rule for interleaving ensures that all continuations of $P \bowtie_L Q$ are of the form $R \bowtie_L Q$ where $P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\, R) > 0$, for some α and $\mathscr{P}$ or of the form $P \bowtie_L R$ where $Q \stackrel{\alpha}{\rightarrowtail} \mathscr{Q}$ and $(\mathscr{Q}\, R) > 0$, for some α and $\mathscr{Q}$. The rule for co-operation, instead, implements the *apparent rate* principle of $PEPA$ (see corollary of Theorem 6): if $\alpha \in L$, $P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}$, $Q \stackrel{\alpha}{\rightarrowtail} \mathscr{Q}$, $(\mathscr{P}\, R_P) = \lambda_P > 0$, and $(\mathscr{Q}\, R_Q) = \lambda_Q > 0$, then $P \bowtie_L Q$ evolves to $R_P \bowtie_L R_Q$ with rate $\frac{\lambda_P}{\oplus\mathscr{P}} \cdot \frac{\lambda_Q}{\oplus\mathscr{Q}} \cdot min\{\oplus\mathscr{P}, \oplus\mathscr{Q}\}$.

The following proposition can be easily proven by derivation induction using Proposition 2(ii) and Proposition 4:

Proposition 7. *For all $P \in \mathcal{P}_{PEPA_k}$, $\alpha \in \mathcal{A}_{PEPA_k}$ and $\mathscr{P} \in \mathcal{P}_{PEPA_k} \to \mathbb{R}_{\geq 0}$, if $P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}$ can be derived from the rules of Fig. 4, then $\mathscr{P} \in \Sigma_{\mathcal{P}_{PEPA_k}}$.* □

Definition 12 (Formal semantics of $PEPA_k$). *The formal semantics of $PEPA_k$ is the RTS $\mathcal{R}_{PEPA_k} =_{\text{def}} (\mathcal{P}_{PEPA_k}, \mathcal{A}_{PEPA_k}, \rightarrowtail)$ where $\rightarrowtail \subseteq \mathcal{P}_{PEPA_k} \times \mathcal{A}_{PEPA_k} \times \Sigma_{\mathcal{P}_{PEPA_k}}$ is the least relation satisfying the rules of Fig. 4.* •

Theorem 6. *$\mathcal{R}_{PEPA_k}$ is total and functional.*

As a corollary of Theorem 6 we get that whenever $P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}$ the apparent rate of α in P—namely the exit rate of P relative to α, denoted by $r_\alpha(P)$ in [18]—is given by $\oplus\mathscr{P}$.

In [9] it is shown that the RTS semantics of $PEPA$ coincides with the original one.

We close this section by observing that $PEPA$ *passive actions* [18] can be easilys dealt with in the RTS approach. One has to consider total functions in $\mathcal{P}_{PEPA_k} \to (\mathbb{R}_{\geq 0} \cup \{w \cdot \top \mid w \in \mathbb{N}_{>0}\})$ and define $\Sigma^{\top}_{\mathcal{P}_{PEPA_k}}$ by restricting only to functions expressed using the $[\ldots]$ notation; all definitions involving $\Sigma_{\mathcal{P}_{PEPA_k}}$ must be extended to $\Sigma^{\top}_{\mathcal{P}_{PEPA_k}}$ accordingly and taking the equations for $\top$ introduced in [18]. The following is an example resulting from the related derivation using the extended definitions:

$$(\alpha, \sqrt{2}).P \bowtie_{\{\alpha\}} ((\alpha, 2\top).Q + (\alpha, 4\top).R) \stackrel{\alpha}{\rightarrowtail} [P \bowtie_{\{\alpha\}} Q \mapsto \frac{\sqrt{2}}{3}, P \bowtie_{\{\alpha\}} R \mapsto \frac{2 \cdot \sqrt{2}}{3}]$$

4.4 CCS-based Stochastic Process Calculi

Our RTS approach has been successfully applied to several CCS-based calculi including Stochastic CCS [19], Stochastic π-calculus [23] and calculi for modeling Service Oriented Computing [8]. The main issue is the treatment of the CCS one-to-one synchronization paradigm, as opposed to the CSP multicast one adopted by $TIPP$, $PEPA$ and $EMPA$. RTS semantics allows for an adequate and elegant calculation of normalization factors which make it possible to preserve nice properties of the original calculi, like associativity of parallel composition, which is not possible using other approaches, as discussed in e.g. [19]. Due to space limitations we do not show the RTS semantics of Stochastic CCS and SOC calculi here and we refer to [8, 9].

5 A Language of Interactive Markov Chains

In this section we show an RTS semantics of Hermanns' Language of Interactive Markov Chains (IML). The definition of Interactive Markov Chains (IMC) follows [14]:

Definition 13. *An Interactive Markov Chain is a tuple $(S, A, \rightarrow, \dashrightarrow, s_0)$ where S is a nonempty, finite set of states, A a finite set of actions, $\rightarrow \subseteq S \times A \times S$ the set of* interactive transitions*, $\dashrightarrow \subseteq S \times \mathbb{R}_{>0} \times S$ the set of* Markov transitions*, and $s_0 \in S$ the initial state.* •

Also for IMCs we let the cumulative transition rate from s to s' be denoted by $\mathbf{rt}(s, s')$. For the sake of simplicity and due to space limitations, in this section we consider a kernel subset IML_k of the language IML defined in [14], which is anyway sufficient for showing how $RTSs$ can be used as a semantic model for IML. In Sect. B.5 we recall the SOS of IML, as defined in [14], for the fragment IML_k we focus on.

5.1 Syntax

The set $\mathcal{P}_{IML_k}$ of IML terms is defined by the following grammar:

$$P ::= \mathbf{0} \;\Big|\; \lambda.P \;\Big|\; a.P \;\Big|\; P+P \;\Big|\; P|[L]|P \;\Big|\; X$$

where $a \in \mathcal{A}_{IML_k}$, with $\mathcal{A}_{IML_k}$ a countable set of *actions*, $\lambda \in \mathbb{R}_{>0}$, finite *synchronisation* set $L \in 2^{\mathcal{A}_{IML_k}}_{fin}$, and X is a constant defined by means of an equation of the form $X \stackrel{\Delta}{=} P$ where constants $X, X_1, X', \ldots$ may occur only guarded in P, i.e. under the scope of a prefix $a.\_$ or $\lambda.\_$.

Semantics In order to give interactive transitions a "first-class objects" status, we consider a slight extension of RTS. We anyway like to point out here that, technically, such an extension is not necessary, as we shall briefly discuss later on. We use it only because it makes our framework closer to the original model of IMCs. The extension of interest, namely RTS^ι, differs from RTS only because, instead of using functions in $\mathcal{P}_{IML_k} \to \mathbb{R}_{\geq 0}$, we consider those in $\mathcal{P}_{IML_k} \to \mathbb{R}^\iota_{\geq 0}$, where $\mathbb{R}^\iota_{\geq 0}$ denotes $(\mathbb{R}_{\geq 0} \cup \{\iota\})$, with ι a distinguished value such that $\iota \notin \mathbb{R}_{\geq 0}$. Markov transitions are modeled as in Sect. 3, using the special element $\surd \notin \mathcal{A}_{IML_k}$ as a label and defining the label set of the relevant RTS^ι as $\mathcal{A}_{IML_k} \cup \{\surd\}$, ranged over by $\alpha, \alpha_1, \alpha', \ldots$. We define $\Sigma^\iota_{\mathcal{P}_{IML_k}}$ as expected:

Definition 14 ($\Sigma^\iota_{\mathcal{P}_{IML_k}}$). *$\Sigma^\iota_{\mathcal{P}_{IML_k}}$ denotes the subset of $\mathcal{P}_{IML_k} \to \mathbb{R}^\iota_{\geq 0}$ including only all functions expressed using the $[\ldots]$ notation, i.e. $\mathscr{P} \in \Sigma^\iota_{\mathcal{P}_{IML_k}}$ if and only if $\mathscr{P} = []$ or $\mathscr{P} = [P_1 \mapsto v_1, \ldots, P_n \mapsto v_n]$ for $n \in \mathbb{N}_{>0}$, $P_1, \ldots, P_n \in \mathcal{P}_{IML_k}$ and $v_1, \ldots, v_n \in \mathbb{R}_{>0} \cup \{\iota\}$, with $([]\,P) =_{\text{def}} 0$ and $[P_1 \mapsto v_1, \ldots, P_n \mapsto v_n]\,P$ yielding v_j if $P = P_j$ for $1 \leq j \leq n$ and 0 otherwise.* •

We extend operations $+$ and $\cdot$ to $+^\iota, \cdot^\iota : \mathbb{R}^\iota_{\geq 0} \to \mathbb{R}^\iota_{\geq 0} \to \mathbb{R}^\iota_{\geq 0}$ as in Fig. 5, where we assume that $v_1, v_2 \notin \{0, \iota\}$. We lift $+^\iota : \mathbb{R}^\iota_{\geq 0} \to \mathbb{R}^\iota_{\geq 0} \to \mathbb{R}^\iota_{\geq 0}$ to $+^\iota$:

$+^\iota$	0	ι	v_2
0	0	ι	v_2
ι	ι	ι	ι
v_1	v_1	ι	$v_1 + v_2$

$\cdot^\iota$	0	ι	v_2
0	0	0	0
ι	0	ι	ι
v_1	0	ι	$v_1 \cdot v_2$

Fig. 5. Definition of $+^\iota$ and $\cdot^\iota$

$\Sigma^{\iota}_{\mathcal{P}_{IML_k}} \to \Sigma^{\iota}_{\mathcal{P}_{IML_k}} \to \mathcal{P}_{IML_k} \to \mathbb{R}^{\iota}_{\geq 0}$; we moreover define $\_|[L]|\_ : \Sigma^{\iota}_{\mathcal{P}_{IML_k}} \to \Sigma^{\iota}_{\mathcal{P}_{IML_k}} \to \mathcal{P}_{IML_k} \to \mathbb{R}^{\iota}_{\geq 0}$ by instantiating $\otimes_L$ on the syntactical constructor for parallel composition on IML_k terms and using $\cdot^{\iota}$ instead of $\cdot$. In the sequel we refrain from using the superscript ι in $+^{\iota}$ and $\cdot^{\iota}$ when it is clear from the context that we are using the extended operators. The following proposition trivially follows from the relevant definitions.

Proposition 8. *(i) All functions in $\Sigma^{\iota}_{\mathcal{P}_{IML_k}}$ yield zero almost everywhere, i.e. for all $\mathscr{P} \in \Sigma^{\iota}_{\mathcal{P}_{IML_k}}$ the set $\{P \in \mathcal{P}_{IML_k} \mid \mathscr{P}\,P \neq 0\}$ is finite; (ii)$\Sigma^{\iota}_{\mathcal{P}_{IML_k}}$ is closed under the extended operators, namely $+, |[L]| : \Sigma^{\iota}_{\mathcal{P}_{IML_k}} \to \Sigma^{\iota}_{\mathcal{P}_{IML_k}} \to \Sigma^{\iota}_{\mathcal{P}_{IML_k}}$.* □

We finally extend the notion of Derived CTMC (see Def. 7) to IMCs in the obvious way:

Definition 15 (Derived IMC). *Let $\mathcal{R} = (S, A, \rightarrowtail)$ be a* functional *RTS^{ι}; for $s_0 \in S$, the IMC of s_0, when one considers only labels in* finite *set $A' \subseteq A$ is defined as $IMC[\{s_0\}, A'] =_{\text{def}} (Der(\{s_0\}, A'), A', \to, \dashrightarrow, s_0)$ where for all $s_1, s_2 \in Der(\{s_0\}, A')$, $\alpha \in A'$ such that $s_1 \overset{\alpha}{\rightarrowtail} \mathscr{P}$:*

- *$s_1 \overset{\alpha}{\to} s_2$ iff $(\mathscr{P}\,s_2) = \iota$;*
- *$s_1 \overset{\lambda}{\dashrightarrow} s_2$ iff $(\mathscr{P}\,s_2) = \lambda > 0$*

$$\frac{}{\mathbf{0} \overset{\alpha}{\rightarrowtail} []} \quad \frac{}{\lambda.P \overset{\checkmark}{\rightarrowtail} [P \mapsto \lambda]} \quad \frac{\alpha \neq \checkmark}{\lambda.P \overset{\alpha}{\rightarrowtail} []} \quad \frac{}{a.P \overset{a}{\rightarrowtail} [P \mapsto \iota]} \quad \frac{\alpha \neq a}{a.P \overset{\alpha}{\rightarrowtail} []}$$

$$\frac{P \overset{\alpha}{\rightarrowtail} \mathscr{P}, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}}{P + Q \overset{\alpha}{\rightarrowtail} \mathscr{P} + \mathscr{Q}} \quad \frac{P \overset{\alpha}{\rightarrowtail} \mathscr{P}, X \overset{\Delta}{=} P}{X \overset{\alpha}{\rightarrowtail} \mathscr{P}}$$

$$\frac{P \overset{\alpha}{\rightarrowtail} \mathscr{P}, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}, \alpha \notin L}{P|[L]|Q \overset{\alpha}{\rightarrowtail} (\mathscr{P}|[L]|(\chi\,Q)) + ((\chi\,P)|[L]|\mathscr{Q})} \quad \frac{P \overset{\alpha}{\rightarrowtail} \mathscr{P}, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}, \alpha \in L}{P|[L]|Q \overset{\alpha}{\rightarrowtail} \mathscr{P}|[L]|\mathscr{Q}}$$

Fig. 6. Rules for the IML_k.

The transition relation $\rightarrowtail$ for IML_k, is defined in Fig. 6. The rule for choice allows for the integration of Markov transitions with interaction ones; as usual, if $P \overset{\checkmark}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\,Q) = \lambda$ then λ is the cumulative rate for reaching Q from P, i.e. $\lambda = \mathbf{rt}(P, Q)$, as can be seen in the derivations in Fig. 7 for term $(\lambda_1.P_1 + a.P_2) + (a.P_2 + \lambda_2.P_1)$.

The rule for interleaving ensures that all continuations of $P\,|[L]|\,Q$ are of the form $R\,|[L]|\,Q$ where $P \overset{\checkmark}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\,R) > 0$ or $P \overset{a}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\,R) = \iota$ for

$$\frac{\dfrac{\dfrac{\lambda_1.P_1 \stackrel{\surd}{\rightarrowtail} [P_1 \mapsto \lambda_1],\ a.P_2 \stackrel{\surd}{\rightarrowtail} []}{\lambda_1.P_1 + a.P_2 \stackrel{\surd}{\rightarrowtail} [P_1 \mapsto \lambda_1]} \qquad \dfrac{a.P_2 \stackrel{\surd}{\rightarrowtail} [],\ \lambda_2.P_1 \stackrel{\surd}{\rightarrowtail} [P_1 \mapsto \lambda_2]}{a.P_2 + \lambda_2.P_1 \stackrel{\surd}{\rightarrowtail} [P_1 \mapsto \lambda_2]}}{(\lambda_1.P_1 + a.P_2) + (a.P_2 + \lambda_2.P_1) \stackrel{\surd}{\rightarrowtail} [P_1 \mapsto \lambda_1] + [P_1 \mapsto \lambda_2]}}{(\lambda_1.P_1 + a.P_2) + (a.P_2 + \lambda_2.P_1) \stackrel{\surd}{\rightarrowtail} [P_1 \mapsto \lambda_1 + \lambda_2]}$$

and similarly

$$\frac{\dfrac{\dfrac{\lambda_1.P_1 \stackrel{a}{\rightarrowtail} [],\ a.P_2 \stackrel{a}{\rightarrowtail} [P_2 \mapsto \iota]}{\lambda_1.P_1 + a.P_2 \stackrel{a}{\rightarrowtail} [P_2 \mapsto \iota]} \qquad \dfrac{a.P_2 \stackrel{a}{\rightarrowtail} [P_2 \mapsto \iota],\ \lambda_2.P_1 \stackrel{a}{\rightarrowtail} []}{a.P_2 + \lambda_2.P_1 \stackrel{a}{\rightarrowtail} [P_2 \mapsto \iota]}}{(\lambda_1.P_1 + a.P_2) + (a.P_2 + \lambda_2.P_1) \stackrel{a}{\rightarrowtail} [P_2 \mapsto \iota] + [P_2 \mapsto \iota]}}{(\lambda_1.P_1 + a.P_2) + (a.P_2 + \lambda_2.P_1) \stackrel{a}{\rightarrowtail} [P_2 \mapsto \iota]}$$

with $(\lambda_1.P_1 + a.P_2) + (a.P_2 + \lambda_2.P_1) \stackrel{\alpha}{\rightarrowtail} []$ for all $\alpha \notin \{a, \surd\}$

Fig. 7. Derivations for $(\lambda_1.P_1 + a.P_2) + (a.P_2 + \lambda_2.P_1)$

some $\mathscr{P}$ and a, or of the form $P \,|[L]|\, R$ where $Q \stackrel{\surd}{\rightarrowtail} \mathscr{Q}$ and $(\mathscr{Q}\, R) > 0$ or $Q \stackrel{a}{\rightarrowtail} \mathscr{Q}$ and $(\mathscr{Q}\, R) = \iota$, for $a \notin L$. The rule for synchronisation, instead, applies only in the case of interactive transitions and postulates that the only terms which can be reached from $P \,|[L]|\, Q$, via $a \in L$ are those of the form $P' \,|[L]|\, Q'$ with $(\mathscr{P}\, P') = (\mathscr{Q}\, Q') = \iota$, where $P \stackrel{a}{\rightarrowtail} \mathscr{P}$ and $Q \stackrel{a}{\rightarrowtail} \mathscr{Q}$.

It is worth noting that we could have chosen to use standard $\Sigma_{\mathcal{P}_{IML_k}}$ instead of its extension $\Sigma^{\iota}_{\mathcal{P}_{IML_k}}$ by replacing axiom $a.P \stackrel{a}{\rightarrowtail} [P \mapsto \iota]$ with $a.P \stackrel{a}{\rightarrowtail} [P \mapsto 1]$. In particular, whenever $P \stackrel{a}{\rightarrowtail} \mathscr{P}$ the number of *different* (interaction) a-transitions from P to Q would be given by $(\mathscr{P}\, Q)$. We preferred the first alternative because we are not interested in counting such transition and we think that keeping different types for the range of the two kinds of transitions makes the framework more clear and closer to the original model of IMCs. We note also a clean separation between internal non-determinism, represented *within* functions, and external non-determinism, represented by different transitions. For instance, assuming P_1, P_2 and P_3 all different terms, the term $a.P_1 + b.P_2 + a.P_3$ has the following transitions:

$$a.P_1 + b.P_2 + a.P_3 \stackrel{a}{\rightarrowtail} [P_1 \mapsto \iota, P_3 \mapsto \iota]$$
$$a.P_1 + b.P_2 + a.P_3 \stackrel{b}{\rightarrowtail} [P_2 \mapsto \iota]$$
$$a.P_1 + b.P_2 + a.P_3 \stackrel{\alpha}{\rightarrowtail} [] \text{ for all } \alpha \notin \{a, b\}.$$

Proposition 9. *For all* $P \in \mathcal{P}_{IML_k}$, $\alpha \in \mathcal{A}_{IML_k} \cup \{\surd\}$ *and* $\mathscr{P} \in \mathcal{P}_{IML_k} \to \mathbb{R}^{\iota}_{\geq 0}$, *if* $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ *can be derived from the rules of Fig. 6, then* $\mathscr{P} \in \Sigma^{\iota}_{\mathcal{P}_{IMC}}$. $\square$

Proposition 10. *For all* $P \in \mathcal{P}_{IML_k}$, $\alpha \in \mathcal{A}_{IML_k} \cup \{\surd\}$ *and* $\mathscr{P} \in \Sigma^{\iota}_{\mathcal{P}_{IMC}}$ *such that* $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ *can be derived from the rules of Fig. 6 the following holds: (i) if* $\alpha \in \mathcal{A}_{IML_k}$ *and* $\mathscr{P} \neq []$ *then* $(\mathsf{range}\,\mathscr{P}) = \{0, \iota\}$, *(ii) if* $\alpha = \surd$ *then* $\iota \notin (\mathsf{range}\,\mathscr{P})$. $\square$.

Definition 16 (Formal semantics of IML_k). *The formal semantics of* IML_k *is the* RTS^{ι} $\mathcal{R}_{IML_k} =_{\text{def}} (\mathcal{P}_{IML_k}, \mathcal{A}_{IML_k} \cup \{\surd\}, \rightarrowtail)$ *where* $\rightarrowtail \subseteq \mathcal{P}_{IML_k} \times (\mathcal{A}_{IML_k} \cup \{\surd\}) \times \Sigma^{\iota}_{\mathcal{P}_{IML_k}}$ *is the least relation satisfying the rules of Fig. 6.* $\bullet$

Theorem 7. $\mathcal{R}_{IML_k}$ *is total and functional.*

Corollary 3. *For all* $P \in \mathcal{P}_{IML_k}, \alpha \in \mathcal{A}_{IML_k} \cup \{\surd\}$ *there exists a* unique $\mathscr{P}$ *such that* $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$.

The following theorem establishes the formal correspondence between the RTS^{ι} semantics of IML_k and the semantics definition given in [14]. Notice that in this case the cumulative rate must be computed over *all* copies of all transitions from P to Q in the *multi*-relation $\dashrightarrow$ defined in [14].

Theorem 8. *For all* $P, Q \in \mathcal{P}_{IML_k}$, $a \in \mathcal{A}_{IML_k}$, *and unique functions* $\mathscr{P}, \mathscr{P}' \in \Sigma_{\mathcal{P}_{IML_k}}$ *such that* $P \overset{a}{\rightarrowtail} \mathscr{P}$ *and* $P \overset{\surd}{\rightarrowtail} \mathscr{P}'$ *the following holds: (i)* $(\mathscr{P}\, Q) = \iota$ *if and only if* $P \overset{a}{\rightarrow} Q$*; (ii)* $(\mathscr{P}'\, Q) = \mathbf{rt}(P, Q)$. $\square$

6 Conclusions

In this paper we presented how *Rate Transition Systems* can be used as a unifying framework for the definition of the semantics of stochastic process algebras. RTSs facilitate the *compositional* definition of such semantics exploiting operators on the next state functions which are the functional counterpart of classical process algebra operators. We applied this framework to representative fragments of major stochastic process calculi including $TIPP$, $EMPA$, $PEPA$ and IML and showed how they solve the issue of transition multiplicity in a simple and elegant way. We, moreover, showed how RTSs help describing different languages, their differences and their similarities. For each calculus, we also proved the formal correspondence between the RTS semantics and the standard SOS one. It turned out that, in most cases, actually in all cases we considered here, it is sufficient to use *functional* RTSs. On the other hand, general RTSs are useful in several situations, as, e.g. in translations of Interactive Markov Chains to Continuous Time Markov Decision Processes (see e.g. [17]), or in the definition of the RTS semantics for the Stochastic π calculus (see [9]). Future work includes the investigation of the nature and actual usefulness of general $RTSs$, and in particular their explicit representation of non-determinism, also in the context of behavioural relations, along the lines of [21].

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A Proof of Proposition 1

Proposition 1. *The transient behaviour of CTMC* $C = (S, \mathbf{R})$ *with* $\mathbf{R}[\bar{s}, \bar{s}] > 0$ *for some* $\bar{s} \in S$ *coincides with that of CTMC* $\tilde{C} = (S, \tilde{\mathbf{R}})$*, such that*

$$\tilde{\mathbf{R}}[s, s'] \ =_{\text{def}} \ \begin{cases} 0 & \textit{if } s = s' \\ \mathbf{R}[s, s'] & \textit{otherwise} \end{cases}$$

□

Proof. Suppose $\mathbf{R}[\bar{s}, \bar{s}] > 0$ and let $(\pi\, \bar{s}\, t)$ be the probability that C is in state $\bar{s}$ at time t, $\mathbb{P}\{C(t) = \bar{s}\}$. For h small enough, the evolution of C in the period $[t, t+h)$ can be captured using $(\pi\, \bar{s}\, t)$ as shown below, letting $p_{\bar{s}}$ denote the probability that no transition from $\bar{s}$ is taken during the period $[t, t+h)$ and $p_{s,\bar{s}}$ denote the probability that a transition from s to $\bar{s}$ takes place during the period $[t, t+h)$[4]:

$\pi\, \bar{s}\, (t + h)$

[4] Notice that, we do *not* require $s \neq \bar{s}$, as usually found in the literature (see, e.g. [13]).

$=$ {Probability Theory; Definition of $p_{\bar{s}}$ and $p_{s,\bar{s}}$; h small}

$(\pi\,\bar{s}\,t)\cdot\left(1-\sum_{s\in S}\mathbf{R}[\bar{s},s]\cdot h\right)+\sum_{s\in S}(\pi\,s\,t)\cdot\mathbf{R}[s,\bar{s}]\cdot h+o(t)$

$=$ {Algebra}

$(\pi\,\bar{s}\,t)-(\pi\,\bar{s}\,t)\cdot\sum_{s\in S\setminus\{\bar{s}\}}\mathbf{R}[\bar{s},s]\cdot h-(\pi\,\bar{s}\,t)\cdot\mathbf{R}[\bar{s},\bar{s}]\cdot h+$

$\sum_{s\in S\setminus\{\bar{s}\}}(\pi\,s\,t)\cdot\mathbf{R}[s,\bar{s}]\cdot h+(\pi\,\bar{s}\,t)\cdot\mathbf{R}[\bar{s},\bar{s}]\cdot h+o(t)$

$=$ {Algebra}

$(\pi\,\bar{s}\,t)-(\pi\,\bar{s}\,t)\cdot\sum_{s\in S\setminus\{\bar{s}\}}\mathbf{R}[\bar{s},s]\cdot h+\sum_{s\in S\setminus\{\bar{s}\}}(\pi\,s\,t)\cdot\mathbf{R}[s,\bar{s}]\cdot h+o(t)$

Thus the evolution of C in the period $[t,t+h)$ does *not* depend on $\mathbf{R}[\bar{s},\bar{s}]$. And in fact, letting

$$\mathbf{Q}_{\mathbf{R}}[s,s'] \;=_{\mathrm{def}}\; \begin{cases}\mathbf{R}[s,s'], \text{if } s\neq s'\\[2ex] -\sum_{s''\in S\setminus\{s\}}\mathbf{R}[s,s''], \text{if } s=s'\end{cases}$$

we get $\pi\,\bar{s}\,(t+h)=(\pi\,\bar{s}\,t)+\left(\sum_{s\in S}(\pi\,s\,t)\cdot\mathbf{Q}_{\mathbf{R}}[s,\bar{s}]\right)\cdot h+o(t)$ from which we get

$$\frac{d(\pi\,\bar{s}\,t)}{dt}=lim_{h\to 0}\frac{(\pi\,\bar{s}\,(t+h))-(\pi\,\bar{s}\,t)}{h}=\sum_{s\in S}\mathbf{Q}_{\mathbf{R}}[s,\bar{s}]\cdot(\pi\,s\,t)$$

The vector $((\pi\,s\,t))_{s\in S}$ of the transient probabilities for C is thus characterised as the solution of the equation

$$\left(\frac{d(\pi\,s\,t)}{dt}\right)_{s\in S}=((\pi\,s\,t))_{s\in S}\mathbf{Q}_{\mathbf{R}} \qquad\qquad \text{given } ((\pi\,s\,0))_{s\in S}$$

which clearly coincides with the equation for the transient probabilities of $\tilde{C}$ observing that $\mathbf{Q}_{\mathbf{R}}=\mathbf{Q}_{\tilde{\mathbf{R}}}$

B SOS definitions

In this section the standard SOS definition of the relevant process calculi is given. We point out that in the literature the transition multi-relation has often be defined as the least multi-relation satisfying a set of SOS rules (see, e.g. [18] or [14]). Although this definition is not completely correct, since the *least* multi-relation happens to be a relation, thus not capturing, as a matter of fact, transition multiplicity. In the sequel we stick to the original formulation for conformance with the original proposals.

B.1 Calculus for finite CTMCs

The SOS of the Calculus for finite CTMCs of Sect. 3 is the multi-LTS $(\mathcal{P}_{CTMC},\mathbb{R}_{>0},\rightarrow)$ where $\rightarrow$ is the multi-relation induced by the rules given in Fig. 8.

$$\frac{}{\lambda.P \xrightarrow{\lambda} P} \quad \frac{P \xrightarrow{\lambda} R}{P+Q \xrightarrow{\lambda} R} \quad \frac{Q \xrightarrow{\lambda} R}{P+Q \xrightarrow{\lambda} R} \quad \frac{P \xrightarrow{\lambda} Q, X \stackrel{\Delta}{=} P}{X \xrightarrow{\lambda} Q}$$

Fig. 8. SOS Rules for the CTMC Calculus.

B.2 $TIPP_k$

The SOS of $TIPP_k$ (see Sect.4.1) is the multi-LTS $(\mathcal{P}_{TIPP_k}, (\mathcal{A}_{TIPP_k} \cup \{\mathtt{i}\}) \times \mathbb{R}_{>0}, \longrightarrow)$ where $\longrightarrow$ is the least multi-relation satisfying the rules given in Fig. 9.

$$\frac{}{(a,\lambda).P \xrightarrow{a,\lambda} P} \quad \frac{P \xrightarrow{a,\lambda} R}{P \,[]\, Q \xrightarrow{a,\lambda} R} \quad \frac{Q \xrightarrow{a,\lambda} R}{P \,[]\, Q \xrightarrow{a,\lambda} R} \quad \frac{P \xrightarrow{a,\lambda} Q, X \stackrel{\Delta}{=} P}{X \xrightarrow{a,\lambda} Q}$$

$$\frac{P \xrightarrow{a,\lambda} P', a \notin L}{P|[L]|Q \xrightarrow{a,\lambda} P'|[L]|Q} \quad \frac{Q \xrightarrow{a,\lambda} Q', a \notin L}{P|[L]|Q \xrightarrow{a,\lambda} P|[L]|Q'}$$

$$\frac{P \xrightarrow{a,\lambda_P} P', Q \xrightarrow{a,\lambda_Q} Q', a \in L}{P|[L]|Q \xrightarrow{a,\lambda_P \cdot \lambda_Q} P'|[L]|Q'}$$

Fig. 9. SOS Rules for the $TIPP_k$.

B.3 $EMPA_k$

The SOS of $EMPA_k$ (see Sect.4.2) is the multi-LTS $(\mathcal{P}_{EMPA_k}, (\mathcal{A}_{EMPA_k} \cup \{\tau\}) \times (\mathbb{R}_{>0} \cup \{*_w \mid w \in \mathbb{R}_{>0}\}), \longrightarrow)$ where $\longrightarrow$ is the multi-relation induced by the rules given in Fig. 10. In the figure, $\tilde{\lambda} \in \mathbb{R}_{>0} \cup \{*_w \mid w \in \mathbb{R}_{>0}\}$, whereas $weight(P,a)$ and $norm(w_1, w_2, a, P_1, P2)$ are defined as follows, where $\sum \{\!|\,|\!\} =_{\text{def}} 0$:

$$weight(P,a) =_{\text{def}} \sum \{\!| w \in \mathbb{R}_{>0} | \exists P' \in \mathcal{P}_{EMPA_k}.\ P \xrightarrow{a,*w} P' |\!\}$$

$$norm(w_1, w_2, a, P_1, P2) =_{\text{def}} \frac{w_1}{weight(P_1,a)} \cdot \frac{w_2}{weight(P_2,a)} \cdot (weight(P_1,a) + weight(P_2,a))$$

B.4 $PEPA_k$

The SOS of $PEPA_k$ (see Sect.4.3) is the multi-LTS $(\mathcal{P}_{PEPA_k}, \mathcal{A}_{PEPA_k} \times \mathbb{R}_{>0}, \longrightarrow)$ where $\longrightarrow$ is the least multi-relation satisfying the rules given in

$$\frac{}{<a,\lambda>.P \xrightarrow{a,\lambda} P} \quad \frac{}{<a,*_w>.P \xrightarrow{a,*w} P} \quad \frac{P \xrightarrow{a,\tilde{\lambda}} R}{P+Q \xrightarrow{a,\tilde{\lambda}} R} \quad \frac{Q \xrightarrow{a,\tilde{\lambda}} R}{P+Q \xrightarrow{a,\tilde{\lambda}} R}$$

$$\frac{P \xrightarrow{a,\tilde{\lambda}} Q, X \stackrel{\Delta}{=} P}{X \xrightarrow{a,\tilde{\lambda}} Q} \quad \frac{P \xrightarrow{a,\tilde{\lambda}} P', a \notin L}{P||_L Q \xrightarrow{a,\tilde{\lambda}} P'||_L Q} \quad \frac{Q \xrightarrow{a,\tilde{\lambda}} Q', a \notin L}{P||_L Q \xrightarrow{a,\tilde{\lambda}} P||_L Q'}$$

$$\frac{P \xrightarrow{a,\lambda} P', Q \xrightarrow{a,*w} Q', a \in L}{P||_L Q \xrightarrow{a,\lambda\cdot\frac{w}{weight(Q,a)}} P'||_L Q'} \quad \frac{P \xrightarrow{a,*w} P', Q \xrightarrow{a,\lambda} Q', a \in L}{P||_L Q \xrightarrow{a,\lambda\cdot\frac{w}{weight(P,a)}} P'||_L Q'}$$

$$\frac{P \xrightarrow{a,*w1} P', Q \xrightarrow{a,*w2} Q', a \in L}{P||_L Q \xrightarrow{a,*norm(w_1,w_2,a,P,Q)} P'||_L Q'}$$

Fig. 10. SOS Rules for the $EMPA_k$.

Fig. 11. In the figure $r_\alpha(P)$ and $r(\alpha, \lambda_1, \lambda_2, P, Q)$ are used, which are defined as follows:

$$r_\alpha((\beta,\lambda).P) \;=_{\text{def}}\; \begin{cases} \lambda, & \text{if } \beta = \alpha \\ 0, & \text{if } \beta \neq \alpha \end{cases}$$

$$r_\alpha(P+Q) \;=_{\text{def}}\; r_\alpha(P) + r_\alpha(Q)$$

$$r_\alpha(P \bowtie_L Q) \;=_{\text{def}}\; \begin{cases} min(r_\alpha(P), r_\alpha(Q)) \text{ if } \alpha \in L \\ r_\alpha(P) + r_\alpha(Q), \text{ if } \alpha \notin L \end{cases}$$

$$r(\alpha, \lambda_1, \lambda_2, P, Q) \;=_{\text{def}}\; \frac{\lambda_1}{r_\alpha(P)} \cdot \frac{\lambda_2}{r_\alpha(Q)} \cdot min(r_\alpha(P), r_\alpha(Q))$$

B.5 IML_k

The SOS definition of IML_k (see Sect.5) is given in Fig. 12. The *action transition* relation $\rightarrow \subset IML_k \times \mathcal{A}_{IML_k} \times IML_k$ is the least relation and the *Markovian transition* relation $\dashrightarrow \subset IML_k \times \mathbb{R}_{>0} \times IML_k$ is the least *multi*-relation given by the rules in Fig. 12. Notice that in [14] parallel composition (and hiding) are not defined by means of an explicit set of SOS rules, but, being derived operators, it is defined by means of expansion laws (and specific laws for hiding). Here we preferred to use explicit SOS rules for uniformity reasons and because we do not address equivalence relations.

$$\frac{}{(\alpha,\lambda).P \xrightarrow{\alpha,\lambda} P{\mapsto}\lambda} \quad \frac{P \xrightarrow{\alpha,\lambda} R}{P{+}Q \xrightarrow{\alpha,\lambda} R} \quad \frac{Q \xrightarrow{\alpha,\lambda} R}{P{+}Q \xrightarrow{\alpha,\lambda} R} \quad \frac{P \xrightarrow{\alpha,\lambda} Q, X \stackrel{\Delta}{=} P}{X \xrightarrow{\alpha,\lambda} Q}$$

$$\frac{P \xrightarrow{\alpha,\lambda} P', \alpha{\notin}L}{P{\bowtie_L}Q \xrightarrow{\alpha,\lambda} P'{\bowtie_L}Q} \quad \frac{Q \xrightarrow{\alpha,\lambda} Q', \alpha{\notin}L}{P{\bowtie_L}Q \xrightarrow{\alpha,\lambda} P{\bowtie_L}Q'}$$

$$\frac{P \xrightarrow{\alpha,\lambda_1} P', Q \xrightarrow{\alpha,\lambda_2} Q', \alpha{\in}L}{P{\bowtie_L}Q \xrightarrow{\alpha,r(\alpha,\lambda_1,\lambda_2,P,Q)} P'{\bowtie_L}Q'}$$

Fig. 11. SOS Rules for the PEPA.

$$\frac{}{a.P \xrightarrow{a} P} \quad \frac{P \xrightarrow{a} R}{P{+}Q \xrightarrow{a} R} \quad \frac{Q \xrightarrow{a} R}{P{+}Q \xrightarrow{a} R} \quad \frac{P \xrightarrow{a} Q, X \stackrel{\Delta}{=} P}{X \xrightarrow{a} Q}$$

$$\frac{P \xrightarrow{a} P', a{\notin}L}{P|[L]|Q \xrightarrow{a} P'|[L]|Q} \quad \frac{Q \xrightarrow{a} Q', a{\notin}L}{P|[L]|Q \xrightarrow{a} P|[L]|Q'} \quad \frac{P \xrightarrow{a} P', Q \xrightarrow{a} Q', a{\in}L}{P|[L]|Q \xrightarrow{a} P'|[L]|Q'}$$

$$\frac{}{\lambda.P \dashrightarrow^{\lambda} P} \quad \frac{P \dashrightarrow^{\lambda} R}{P{+}Q \dashrightarrow^{\lambda} R} \quad \frac{Q \dashrightarrow^{\lambda} R}{P{+}Q \dashrightarrow^{\lambda} R} \quad \frac{P \dashrightarrow^{\lambda} Q, X \stackrel{\Delta}{=} P}{X \dashrightarrow^{\lambda} Q}$$

$$\frac{P \dashrightarrow^{\lambda} P'}{P|[L]|Q \dashrightarrow^{\lambda} P'|[L]|Q} \quad \frac{Q \dashrightarrow^{\lambda} Q'}{P|[L]|Q \dashrightarrow^{\lambda} P|[L]|Q'}$$

Fig. 12. SOS Rules for the IML_k.

C Proofs related to Sect. 3

C.1 Proof of Proposition 3

Proposition 3. For all $P \in \mathcal{P}_{CTMC}$ and $\mathscr{P} \in \mathcal{P}_{CTMC} \to \mathbb{R}_{\geq 0}$, if $P \rightarrowtail \mathscr{P}$ can be derived from the rules of Fig. 1, then $\mathscr{P} \in \Sigma_{\mathcal{P}_{CTMC}}$. □

Proof. By derivation induction. Let $n \geq 1$ be the length of the derivation for proving $P \rightarrowtail \mathscr{P}$.
Base case: Trivial since the only cases in which $P \rightarrowtail \mathscr{P}$ can be derived with a proof of length 1 are those in which $\mathscr{P} = []$ or $\mathscr{P} = [P' \mapsto \lambda]$ and $[], [P' \mapsto \lambda] \in \Sigma_{\mathcal{P}_{CTMC}}$ by definition.
Inductive step: The last assert of any proof of length $n > 1$ must be of the form $P + Q \rightarrowtail \mathscr{P} + \mathscr{Q}$ or $X \rightarrowtail \mathscr{P}$. In the first case $\mathscr{P} + \mathscr{Q} \in \Sigma_{\mathcal{P}_{CTMC}}$, using Proposition 2 since $\mathscr{P}, \mathscr{Q} \in \Sigma_{\mathcal{P}_{CTMC}}$ by I.H. In the second case the assert trivially follows from the I.H.

C.2 Proof of Theorem 1

Theorem 1. $\mathcal{R}_{CTMC}$ is total and functional. □

Proof. $\mathcal{R}_{CTMC}$ *is total*: By induction on the structure, taking inaction and prefix as base cases, for which the assert is trivially proven. For the inductive step we show only the case $P + Q$ which is also very simple because $P \overset{\checkmark}{\rightarrowtail} \mathscr{P}$ and $Q \overset{\checkmark}{\rightarrowtail} \mathscr{Q}$, for some $\mathscr{P}$ and $\mathscr{Q}$ by the I.H., hence $P + Q \overset{\checkmark}{\rightarrowtail} \mathscr{P} + \mathscr{Q}$ by the *RTS* semantics of the CTMC Language (Fig. 1).

$\mathcal{R}_{CTMC}$ *is functional*: By induction on the length of the derivation. We prove only the inductive step for case $P + Q$ here, the others being simpler.

$P + Q \overset{\checkmark}{\rightarrowtail} \mathscr{R}_1,\ P + Q \overset{\checkmark}{\rightarrowtail} \mathscr{R}_2$

$\Rightarrow$ {Def. of $\rightarrowtail$ (Fig. 1)}

$\mathscr{R}_1 = \mathscr{P}_1 + \mathscr{Q}_1, \mathscr{R}_2 = \mathscr{P}_2 + \mathscr{Q}_2, P \overset{\checkmark}{\rightarrowtail} \mathscr{P}_1, Q \overset{\checkmark}{\rightarrowtail} \mathscr{Q}_1, P \overset{\checkmark}{\rightarrowtail} \mathscr{P}_2, Q \overset{\checkmark}{\rightarrowtail} \mathscr{Q}_2$

$\Rightarrow$ {I.H.}

$\mathscr{P}_1 = \mathscr{P}_2,\ \mathscr{Q}_1 = \mathscr{Q}_2,\ \mathscr{R}_1 = \mathscr{P}_1 + \mathscr{Q}_1, \mathscr{R}_2 = \mathscr{P}_2 + \mathscr{Q}_2$

$\Rightarrow$ {Algebra}

$\mathscr{R}_1 = \mathscr{R}_2$

D Proofs related to Sect. 4

D.1 Proof of Proposition 5

Proposition 5. For all $P \in \mathcal{P}_{TIPP_k}$, $a \in \mathcal{A}_{TIPP_k} \cup \{\mathtt{i}\}$ and $\mathscr{P} \in \mathcal{P}_{TIPP_k} \to \mathbb{R}_{\geq 0}$, if $P \stackrel{a}{\rightarrowtail} \mathscr{P}$ can be derived from the rules of Fig. 2, then $\mathscr{P} \in \Sigma_{\mathcal{P}_{TIPP_k}}$. □

Proof. By induction on the length of the derivation for $P \stackrel{a}{\rightarrowtail} \mathscr{P}$. We prove only the inductive step. The last assert of any proof of length $n > 1$ must be of the form $P[]Q \rightarrowtail \mathscr{P} + \mathscr{Q}$, or $P|[L]|Q \stackrel{a}{\rightarrowtail} (\mathscr{P}|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q})$, or $P|[L]|Q \stackrel{a}{\rightarrowtail} \mathscr{P}|[L]|\mathscr{Q}$, or $X \rightarrowtail \mathscr{P}$. In all cases the assert follows using Proposition 2 and Proposition 4 since $\mathscr{P}, \mathscr{Q} \in \Sigma_{\mathcal{P}_{TIPP_k}}$ by I.H. and $(\chi\, P), (\chi\, Q) \in \Sigma_{\mathcal{P}_{TIPP_k}}$ by definition.

D.2 Proof of Theorem 2

Theorem 2. $\mathcal{R}_{TIPP_k}$ is total and functional. □

Proof. $\mathcal{R}_{TIPP_k}$ is total: By induction on the structure, taking inaction and prefix as base cases, for which the assert is trivially proven. For the inductive step we show only the case $P\,|[L]|\,Q$ which is also very simple because $P \stackrel{a}{\rightarrowtail} \mathscr{P}$ and $Q \stackrel{a}{\rightarrowtail} \mathscr{Q}$, for some $\mathscr{P}$ and $\mathscr{Q}$ by the I.H., hence, assuming $a \notin L$ $P\,|[L]|\,Q \stackrel{a}{\rightarrowtail} (\mathscr{P}|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q})$ by the *RTS* semantics of $TIPP_k$ (Fig. 2); the case for $a \in L$ is similar.

$\mathcal{R}_{TIPP_k}$ *is functional*: By induction of the length of the derivation for $P \stackrel{a}{\rightarrowtail} \mathscr{P}$. We prove only the inductive step for case $P\,|[L]|\,Q$ here, the others being similar or simpler. Let us suppose there are two different derivations of length $n > 1$: $P\,|[L]|\,Q \stackrel{a}{\rightarrowtail} \mathscr{R}_1$ and $P\,|[L]|\,Q \stackrel{a}{\rightarrowtail} \mathscr{R}_2$, with $a \notin L$:

$P\,|[L]|\,Q \stackrel{a}{\rightarrowtail} \mathscr{R}_1,\ P\,|[L]|\,Q \stackrel{a}{\rightarrowtail} \mathscr{R}_2$

$\Rightarrow$ $\{$Def. of $\rightarrowtail$ (Fig. 2)$\}$

$\mathscr{R}_1 = (\mathscr{P}_1|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q}_1), \mathscr{R}_2 = (\mathscr{P}_2|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q}_2)$

$\Rightarrow$ $\{$I.H.$\}$

$\mathscr{P}_1 = \mathscr{P}_2, \mathscr{Q}_1 = \mathscr{Q}_2,$

$\mathscr{R}_1 = (\mathscr{P}_1|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q}_1), \mathscr{R}_2 = (\mathscr{P}_2|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q}_2)$

$\Rightarrow$ $\{$Algebra$\}$

$\mathscr{R}_1 = \mathscr{R}_2$

The case $P\,|[L]|\,Q \stackrel{a}{\rightarrowtail} \mathscr{R}_1$ and $P\,|[L]|\,Q \stackrel{a}{\rightarrowtail} \mathscr{R}_2$, with $a \in L$ is similar.

D.3 Proof of Theorem 3

Theorem 3. For all $P, Q \in \mathcal{P}_{TIPP_k}$, $a \in \mathcal{A}_{TIPP_k} \cup \{\mathtt{i}\}$, and unique $\mathscr{P} \in \Sigma_{\mathcal{P}_{TIPP_k}}$ such that $P \stackrel{a}{\rightarrowtail} \mathscr{P}$ the following holds: $(\mathscr{P}\,Q) = \mathbf{rt}_a(P,Q)$ □

Proof. By induction of the length of the derivation for $P \stackrel{a}{\rightarrowtail} \mathscr{P}$. We prove only the inductive step for case $P_1|[L]|P_2$, under the assumption $a \in L$, the other cases being similar. By definition of the RTS semantics of $TIPP_k$, the last assert of the derivation is of the form $P_1|[L]|P_2 \stackrel{a}{\rightarrowtail} \mathscr{P}_1|[L]|\mathscr{P}_2$, with $P_1 \stackrel{a}{\rightarrowtail} \mathscr{P}_1$ and $P_2 \stackrel{a}{\rightarrowtail} \mathscr{P}_2$. We observe that if Q is not of the form $Q_1|[L]|Q_2$ then $(\mathscr{P}_1|[L]|\mathscr{P}_2)\,Q = 0$. On the other hand, we observe that the only transitions from $P_1|[L]|P_2$ allowed by the SOS semantics of $TIPP_k$ are to terms of the form $Q_1|[L]|Q_2$, so also $\mathbf{rt}_a(P_1|[L]|P_2, Q) = 0$ if Q is not of the form $Q_1|[L]|Q_2$. Let us assume Q is of the form $Q_1|[L]|Q_2$.

$$
\begin{array}{ll}
 & (\mathscr{P}_1|[L]|\mathscr{P}_2)\,Q_1|[L]|Q_2 \\
= & \quad \{\text{Def. } (\mathscr{P}_1|[L]|\mathscr{P}_2)\} \\
 & (\mathscr{P}_1\,Q_1)\cdot(\mathscr{P}_2\,Q_2) \\
= & \quad \{P_1 \stackrel{a}{\rightarrowtail} \mathscr{P}_1 \text{ and } P_2 \stackrel{a}{\rightarrowtail} \mathscr{P}_2;\ \text{I.H.}\} \\
 & \mathbf{rt}_a(P_1,Q_1)\cdot\mathbf{rt}_a(P_2,Q_2) \\
= & \quad \{\text{SOS definition of } TIPP_k;\ \text{Def. of } \mathbf{rt}_a\} \\
 & \mathbf{rt}_a(P_1|[L]|P_2, Q_1|[L]|Q_2)
\end{array}
$$

D.4 Proof of Proposition 6

Proposition 6. For all $P \in \mathcal{P}_{EMPA_k}$, $\alpha \in \mathcal{A}_{EMPA_k*} \cup \{\tau\}$ and $\mathscr{P} \in \mathcal{P}_{EMPA_k} \to \mathbb{R}_{\geq 0}$, if $P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}$ can be derived from the rules of Fig. 3, then $\mathscr{P} \in \Sigma_{\mathcal{P}_{EMPA_k}}$.□

Proof. By derivation induction. Let $n \geq 1$ be the length of the derivation for proving $P \stackrel{a}{\rightarrowtail} \mathscr{P}$.
Base case: Trivial since the only cases in which $P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}$ can be derived with a proof of length 1 are those in which $\mathscr{P} = []$ or $\mathscr{P} = [P' \mapsto x]$, with $x \in \mathbb{R}_{>0}$, and $[], [P' \mapsto x] \in \Sigma_{\mathcal{P}_{EMPA_k}}$ by definition.
Inductive step: The last assert of any proof of length $n > 1$ must be of the form $P + Q \stackrel{\alpha}{\rightarrowtail} \mathscr{P} + \mathscr{Q}$, or $P||_L Q \stackrel{\alpha}{\rightarrowtail} (\mathscr{P}||_L(\chi\,Q)) + ((\chi\,P)||_L\mathscr{Q})$, or $P||_L Q \stackrel{a*}{\rightarrowtail} \mathscr{P}||_L\mathscr{Q}\cdot\frac{(\oplus\mathscr{P})+(\oplus\mathscr{Q})}{(\oplus\mathscr{P})\cdot(\oplus\mathscr{Q})}$, or $P||_L Q \stackrel{a}{\rightarrowtail} \mathscr{P}_o||_L\mathscr{Q}_i\cdot\frac{1}{\oplus\mathscr{Q}_i} + \mathscr{P}_i||_L\mathscr{Q}_o\cdot\frac{1}{\oplus\mathscr{P}_i}$, or, finally $X \rightarrowtail \mathscr{P}$. In all cases the assert follows using Proposition 2 and Proposition 4 since $\mathscr{P}, \mathscr{Q} \in \Sigma_{\mathcal{P}_{EMPA_k}}$ by I.H. and $(\chi\,P), (\chi\,Q) \in \Sigma_{\mathcal{P}_{EMPA_k}}$ by definition.

D.5 Proof of Theorem 4

Theorem 4. $\mathcal{R}_{EMPA_k}$ is total and functional. □

Proof. $\mathcal{R}_{EMPA_k}$ *is total*: By induction on the structure, taking inaction and prefix as base cases, for which the assert is trivially proven. For the inductive step we show only the case $P \,||_L\, Q$ which is also very simple because $P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}$ and $Q \stackrel{\alpha}{\rightarrowtail} \mathscr{Q}$, for some $\mathscr{P}$ and $\mathscr{Q}$ by the I.H., hence, assuming $(n\,\alpha) \notin L$ $P \,||_L\, Q \stackrel{\alpha}{\rightarrowtail} (\mathscr{P}||_L(\chi\, Q)) \,+\, ((\chi\, P)||_L\mathscr{Q})$ by the RTS semantics of $EMPA_k$ (Fig. 3). The case for $(n\,\alpha) \in L$ is similar.

$\mathcal{R}_{EMPA_k}$ *is functional*: By induction of the length on the derivation for $P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}$. We prove only the inductive step for case $P \,||_L\, Q$ here, the others being similar or simpler. Let us suppose $P \,||_L\, Q \stackrel{\alpha}{\rightarrowtail} \mathscr{R}_1$ and $P \,||_L\, Q \stackrel{\alpha}{\rightarrowtail} \mathscr{R}_2$, with $(n\,\alpha) \notin L$:

$$P \,||_L\, Q \stackrel{\alpha}{\rightarrowtail} \mathscr{R}_1,\, P \,||_L\, Q \stackrel{\alpha}{\rightarrowtail} \mathscr{R}_2$$

$\Rightarrow \quad$ {Def. of $\rightarrowtail$ (Fig. 3)}

$$\mathscr{R}_1 = (\mathscr{P}_1||_L(\chi\, Q)) \,+ ((\chi\, P)||_L\mathscr{Q}_1), \mathscr{R}_2 = (\mathscr{P}_2||_L(\chi\, Q)) \,+ ((\chi\, P)||_L\mathscr{Q}_2)$$

$\Rightarrow \quad$ {I.H.}

$$\mathscr{P}_1 = \mathscr{P}_2, \mathscr{Q}_1 = \mathscr{Q}_2,$$

$$\mathscr{R}_1 = (\mathscr{P}_1||_L(\chi\, Q)) \,+ ((\chi\, P)||_L\mathscr{Q}_1), \mathscr{R}_2 = (\mathscr{P}_2||_L(\chi\, Q)) \,+ ((\chi\, P)||_L\mathscr{Q}_2)$$

$\Rightarrow \quad$ {Algebra}

$$\mathscr{R}_1 = \mathscr{R}_2$$

The case $P \,||_L\, Q \stackrel{a*}{\rightarrowtail} \mathscr{R}_1$ and $P \,||_L\, Q \stackrel{a*}{\rightarrowtail} \mathscr{R}_2$ with $a \in L$ is similar as well as case $P \,||_L\, Q \stackrel{a}{\rightarrowtail} \mathscr{R}_1$ and $P \,||_L\, Q \stackrel{a}{\rightarrowtail} \mathscr{R}_2$ with $a \in L$.

D.6 Proof of Theorem 5

Theorem 5. For all $P, Q \in \mathcal{P}_{EMPA_k}$, $a, a* \in \mathcal{A}_{EMPA_{k*}}$ or $a = \tau$, and unique functions $\mathscr{P}, \mathscr{P}^* \in \Sigma_{\mathcal{P}_{EMPA_k}}$ such that $P \stackrel{a}{\rightarrowtail} \mathscr{P}$ and $P \stackrel{a*}{\rightarrowtail} \mathscr{P}^*$ the following holds: $(\mathscr{P}\, Q) = \mathbf{rt}_a(P, Q)$, $(\mathscr{P}^*\, Q) = \mathbf{wt}_a(P, Q)$, and $(\oplus\mathscr{P}^*) = weight(P, a)$ □

Proof. We prove the assert by induction on the length of the derivations for $P \stackrel{a}{\rightarrowtail} \mathscr{P}$ and for $P \stackrel{a*}{\rightarrowtail} \mathscr{P}'$. We prove only the inductive step for case $P_1||_LP_2$, under the assumption $a \in L$, the other cases being similar. By definition of the RTS semantics of $EMPA_k$, the last asserts of the derivations are of the form $P_1||_LP_2 \stackrel{a}{\rightarrowtail} \mathscr{P}_1^0||_L\mathscr{P}_2^i \cdot \frac{1}{\oplus\mathscr{P}_2^i} \,+\, \mathscr{P}_1^i||_L\mathscr{P}_2^o \cdot \frac{1}{\oplus\mathscr{P}_1^i}$ and

$P_1||_L P_2 \stackrel{a*}{\rightarrowtail} \mathscr{P}_1||_L \mathscr{P}_2 \cdot \frac{(\oplus \mathscr{P}_1)+(\oplus \mathscr{P}_2)}{(\oplus \mathscr{P}_1)\cdot(\oplus \mathscr{P}_2)}$. We observe that if Q is not of the form $Q_1||_L Q_2$ then $(\mathscr{P}_1^o||_L \mathscr{P}_2^i)\, Q = (\mathscr{P}_1^i||_L \mathscr{P}_2^o)\, Q = (\mathscr{P}_1||_L \mathscr{P}_2)\, Q = 0$. On the other hand, we observe that the only transitions from $P_1||_L P_2$ allowed by the SOS semantics of $EMPA_k$ are to terms of the form $Q_1||_L Q_2$, so also $\mathbf{rt}_a(P_1||_L P_2, Q) = \mathbf{wt}_a(P_1||_L P_2, Q) = 0$ if Q is not of the form $Q_1||_L Q_2$. Let us assume Q is of the form $Q_1||_L Q_2$.

$$\begin{cases} (\mathscr{P}_1^o||_L \mathscr{P}_2^i \cdot \frac{1}{\oplus \mathscr{P}_2^i} + \mathscr{P}_1^i||_L \mathscr{P}_2^o \cdot \frac{1}{\oplus \mathscr{P}_1^i})\, Q_1||_L Q_2 \\ (\mathscr{P}_1||_L \mathscr{P}_2 \cdot \frac{(\oplus \mathscr{P}_1)+(\oplus \mathscr{P}_2)}{(\oplus \mathscr{P}_1)\cdot(\oplus \mathscr{P}_2)})\; Q_1||_L Q_2 \\ \oplus\left((\mathscr{P}_1||_L \mathscr{P}_2) \cdot \frac{(\oplus \mathscr{P}_1)+(\oplus \mathscr{P}_2)}{(\oplus \mathscr{P}_1)\cdot(\oplus \mathscr{P}_2)}\right) \end{cases}$$

$$= \qquad \{\text{Def. } ||_L, \oplus\}$$

$$\begin{cases} (\mathscr{P}_1^o\, Q_1) \cdot (\mathscr{P}_2^i\, Q_2) \cdot \frac{1}{\oplus \mathscr{P}_2^i} + (\mathscr{P}_1^i\, Q_1) \cdot (\mathscr{P}_2^o\, Q_2) \cdot \frac{1}{\oplus \mathscr{P}_1^i} \\ (\mathscr{P}_1\, Q_1) \cdot (\mathscr{P}_2\, Q_2) \cdot \frac{(\oplus \mathscr{P}_1)+(\oplus \mathscr{P}_2)}{(\oplus \mathscr{P}_1)\cdot(\oplus \mathscr{P}_2)} \\ \frac{(\oplus \mathscr{P}_1)+(\oplus \mathscr{P}_2)}{(\oplus \mathscr{P}_1)\cdot(\oplus \mathscr{P}_2)} \cdot \sum_{Q_1'||_L Q_2' \in \mathcal{P}_{EMPA_k}} (\mathscr{P}_1\, Q_1') \cdot (\mathscr{P}_2\, Q_2') \end{cases}$$

$$= \qquad \{P_h \stackrel{a*}{\rightarrowtail} \mathscr{P}_h^i, P_h \stackrel{a*}{\rightarrowtail} \mathscr{P}_h, h = 1,2;\ \text{Unicity theorem (Th. 4)}\}$$

$$\begin{cases} (\mathscr{P}_1^o\, Q_1) \cdot (\mathscr{P}_2^i\, Q_2) \cdot \frac{1}{\oplus \mathscr{P}_2^i} + (\mathscr{P}_1^i\, Q_1) \cdot (\mathscr{P}_2^o\, Q_2) \cdot \frac{1}{\oplus \mathscr{P}_1^i} \\ (\mathscr{P}_1^i\, Q_1) \cdot (\mathscr{P}_2^i\, Q_2) \cdot \frac{(\oplus \mathscr{P}_1^i)+(\oplus \mathscr{P}_2^i)}{(\oplus \mathscr{P}_1^i)\cdot(\oplus \mathscr{P}_2^i)} \\ \frac{(\oplus \mathscr{P}_1^i)+(\oplus \mathscr{P}_2^i)}{(\oplus \mathscr{P}_1^i)\cdot(\oplus \mathscr{P}_2^i)} \cdot \sum_{Q_1'||_L Q_2' \in \mathcal{P}_{EMPA_k}} (\mathscr{P}_1^i\, Q_1') \cdot (\mathscr{P}_2^i\, Q_2') \end{cases}$$

$$= \qquad \{P_h \stackrel{a}{\rightarrowtail} \mathscr{P}_h^o, P_h \stackrel{a*}{\rightarrowtail} \mathscr{P}_h^i, h = 1,2;\ \text{I.H.}\}$$

$$\begin{cases} \frac{\mathbf{rt}_a(P_1,Q_1)\cdot \mathbf{wt}_a(P_2,Q_2)}{weight(a,P_2)} + \frac{\mathbf{wt}_a(P_1,Q_1)\cdot \mathbf{rt}_a(P_2,Q_2)}{weight(a,P_1)} \\ \mathbf{wt}_a(P_1, Q_1) \cdot \mathbf{wt}_a(P_2, Q_2) \cdot \frac{weight(a,P_1)+weight(a,P_2)}{weight(a,P_1)\cdot weight(a,P_2)} \\ \frac{weight(a,P_1)+weight(a,P_2)}{weight(a,P_1)\cdot weight(a,P_2)} \cdot \sum_{Q_1'||_L Q_2' \in \mathcal{P}_{EMPA_k}} \mathbf{wt}_a(P_1, Q_1') \cdot \mathbf{wt}_a(P_2\, Q_2') \end{cases}$$

$$= \qquad \{\text{Def. } \mathbf{rt}_a;\ \text{Def. } \mathbf{wt}_a;\ \text{Def. } weight;\ \text{SOS definition of } EMPA_k\}$$

$$\begin{cases} \mathbf{rt}_a(P_1||_L P_2, Q_1||_L Q_2) \\ \mathbf{wt}_a(P_1||_L P_2, Q_1||_L Q_2) \\ weight(a, P_1||_L P_2) \end{cases}$$

E Proofs related to Sect. 5

E.1 Proof of Proposition 9

Proposition 9. For all $P \in \mathcal{P}_{IML_k}$, $\alpha \in \mathcal{A}_{IML_k} \cup \{\surd\}$ and $\mathscr{P} \in \mathcal{P}_{IML_k} \to \mathbb{R}^{\iota}_{\geq 0}$, if $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ can be derived from the rules of Fig. 6, then $\mathscr{P} \in \Sigma^{\iota}_{\mathcal{P}_{IMC}}$. □

Proof. By induction on the length of the derivation for $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$. We prove only the inductive step. The last assert of any proof of length $n > 1$ must be of the form $P + Q \rightarrowtail \mathscr{P} + \mathscr{Q}$, or $P|[L]|Q \overset{\alpha}{\rightarrowtail} (\mathscr{P}|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q})$, or $P|[L]|Q \overset{a}{\rightarrowtail} \mathscr{P}|[L]|\mathscr{Q}$, or $X \overset{\alpha}{\rightarrowtail} \mathscr{P}$. In all cases the assert follows using Proposition 8 since $\mathscr{P}, \mathscr{Q} \in \Sigma^{\iota}_{\mathcal{P}_{IMC}}$ by I.H. and $(\chi\, P), (\chi\, Q) \in \Sigma^{\iota}_{\mathcal{P}_{IMC}}$ by definition.

E.2 Proof of Proposition 10

Proposition 10. For all $P \in \mathcal{P}_{IML_k}$, $\alpha \in \mathcal{A}_{IML_k} \cup \{\surd\}$ and $\mathscr{P} \in \Sigma^{\iota}_{\mathcal{P}_{IMC}}$ such that $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ can be derived from the rules of Fig. 6 the following holds: (i) if $\alpha \in \mathcal{A}_{IML_k}$ and $\mathscr{P} \neq []$ then $(\mathsf{range}\, \mathscr{P}) = \{0, \iota\}$, (ii) if $\alpha = \surd$ then $\iota \notin (\mathsf{range}\, \mathscr{P})$. □.

Proof. By induction on the length of the derivation for $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$. Let $n \geq 1$ be the length of the derivation for proving $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$.
Base case: Trivial since the only cases in which $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ can be derived with a proof of length 1 are those in which $\mathscr{P} = []$ or $\lambda.R \overset{\surd}{\rightarrowtail} \mathscr{P} = [R \mapsto \lambda]$, or $a.R \overset{a}{\rightarrowtail} \mathscr{P} = [R \mapsto \iota]$, for $a \in \mathcal{A}_{IML_k}$. In all these cases the assert easily follows from the definition of $[R \mapsto v]$.
Inductive step: The last assert of any proof of length $n > 1$ must be of the form $P + Q \overset{\alpha}{\rightarrowtail} \mathscr{P} + \mathscr{Q}$, or, $X \overset{\alpha}{\rightarrowtail} \mathscr{P}$, or $P|[L]|Q \overset{\alpha}{\rightarrowtail} (\mathscr{P}|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q})$, or $P|[L]|Q \overset{\alpha}{\rightarrowtail} \mathscr{P}|[L]|\mathscr{Q}$.

Case: $P + Q \overset{\alpha}{\rightarrowtail} \mathscr{P} +^{\iota} \mathscr{Q}$

(i) Suppose $\alpha = a \in \mathcal{A}_{IML_k}$, $\mathscr{P} + \mathscr{Q} \neq []$; from the definition of the RTS semantics of IML_k and of $+^{\iota}$ we get $P \overset{a}{\rightarrowtail} \mathscr{P}$, $Q \overset{a}{\rightarrowtail} \mathscr{Q}$ and $\mathscr{P} \neq []$ or $\mathscr{Q} \neq []$. If $\mathscr{P} \neq []$ and $\mathscr{Q} = []$ we proceed with the following derivation (the case $\mathscr{P} = []$ and $\mathscr{Q} \neq []$ is similar):

$P \overset{a}{\rightarrowtail} \mathscr{P}, \mathscr{P} \neq [], \mathscr{Q} = [],$

$\Rightarrow \quad \{\text{I.H.}\}$

$(\mathsf{range}\, \mathscr{P}) = \{0, \iota\}, \mathscr{Q} = []$

$\Rightarrow \quad \{\text{Def. } +^{\iota}\text{; Def. []; Def. } \mathsf{range}\}$

$\mathsf{range}(\mathscr{P} +^{\iota} \mathscr{Q}) = \{0, \iota\}$

If $\mathscr{P} \neq []$ and $\mathscr{Q} \neq []$ we proceed with the following derivation:

$P \overset{a}{\rightarrowtail} \mathscr{P}, Q \overset{a}{\rightarrowtail} \mathscr{Q}, \mathscr{P} \neq [], \mathscr{Q} \neq [],$

$\Rightarrow \quad \{\text{I.H.}\}$

$(\mathsf{range}\, \mathscr{P}) = (\mathsf{range}\, \mathscr{Q}) = \{0, \iota\}$

$\Rightarrow \quad \{\text{Def. } +^{\iota}\text{; Def. } \mathsf{range}\text{; Prop. 8(i)}\}$

$\mathsf{range}(\mathscr{P} +^{\iota} \mathscr{Q}) = \{0, \iota\}$

(ii) Suppose $\alpha = \surd$:

$P + Q \overset{\surd}{\rightarrowtail} \mathscr{P} + \mathscr{Q}$

$\Rightarrow \quad \{\text{Def. } RTS \text{ semantics of } IML_k\}$

$P \overset{\surd}{\rightarrowtail} \mathscr{P}, Q \overset{\surd}{\rightarrowtail} \mathscr{Q}$

$\Rightarrow \quad \{\text{I.H.}\}$

$\iota \notin (\mathsf{range}\, \mathscr{P}), \iota \notin (\mathsf{range}\, \mathscr{Q})$

$\Rightarrow \quad \{\text{Def. } +^{\iota}\}$

$\iota \notin \mathsf{range}(\mathscr{P} + \mathscr{Q})$

Case: $X \overset{\alpha}{\rightarrowtail} \mathscr{P}, X \triangleq R$

(i) Suppose $\alpha = a \in \mathcal{A}_{IML_k}$, $\mathscr{P} \neq []$:

$X \overset{a}{\rightarrowtail} \mathscr{P}, \mathscr{P} \neq [], X \triangleq R$

$\Rightarrow \quad \{\text{Def. } RTS \text{ semantics of } IML_k\}$

$R \overset{a}{\rightarrowtail} \mathscr{P}, \mathscr{P} \neq []$

$\Rightarrow \quad \{\text{I.H.}\}$

$(\mathsf{range}\, \mathscr{P}) = \{0, \iota\}$

(ii) Suppose $\alpha = \surd$:

$$X \overset{\surd}{\rightarrowtail} \mathscr{P}, X \triangleq R$$

$\Rightarrow$ $\{$Def. RTS semantics of $IML_k\}$

$$R \overset{\surd}{\rightarrowtail} \mathscr{P}$$

$\Rightarrow$ $\{$I.H.$\}$

$$\iota \notin (\mathsf{range}\, \mathscr{P})$$

Case: $P|[L]|Q \overset{\alpha}{\rightarrowtail} (\mathscr{P}|[L]|(\chi\, Q)) +^{\iota} ((\chi\, P)|[L]|\mathscr{Q})$

(i) Suppose $\alpha = a \in \mathcal{A}_{IML_k}$, $(\mathscr{P}|[L]|(\chi\, Q)) +^{\iota} ((\chi\, P)|[L]|\mathscr{Q}) \neq []$: from the definition of the RTS semantics of IML_k and of $+^{\iota}$ we get $P \overset{a}{\rightarrowtail} \mathscr{P}$, $Q \overset{a}{\rightarrowtail} \mathscr{Q}$, $(\mathscr{P}|[L]|(\chi\, Q)) \neq []$ or $((\chi\, P)|[L]|\mathscr{Q}) \neq []$. If $(\mathscr{P}|[L]|(\chi\, Q)) \neq []$ and $((\chi\, P)|[L]|\mathscr{Q}) = []$ we proceed with the following derivation (the case $(\mathscr{P}|[L]|(\chi\, Q)) = []$ and $((\chi\, P)|[L]|\mathscr{Q}) \neq []$ is similar)

$$P \overset{a}{\rightarrowtail} \mathscr{P}, (\mathscr{P}|[L]|(\chi\, Q)) \neq [], ((\chi\, P)|[L]|\mathscr{Q}) = []$$

$\Rightarrow$ $\{$Def. $|[L]|\}$

$$P \overset{a}{\rightarrowtail} \mathscr{P}, \mathscr{P} \neq [], ((\chi\, P)|[L]|\mathscr{Q}) = []$$

$\Rightarrow$ $\{$I.H.$\}$

$$(\mathsf{range}\, \mathscr{P}) = \{0, \iota\}, ((\chi\, P)|[L]|\mathscr{Q}) = []$$

$\Rightarrow$ $\{$Def. $|[L]|\}$

$$(\mathsf{range}\, (\mathscr{P}|[L]|(\chi\, Q))) = \{0, \iota\}, ((\chi\, P)|[L]|\mathscr{Q}) = []$$

$\Rightarrow$ $\{$Def. $+^{\iota}$; Def. $[]$; Def. $\mathsf{range}\}$

$$\mathsf{range}((\mathscr{P}|[L]|(\chi\, Q))) + ((\chi\, P)|[L]|\mathscr{Q})) = \{0, \iota\}$$

If $(\mathscr{P}|[L]|(\chi\, Q)) \neq []$ and $((\chi\, P)|[L]|\mathscr{Q}) \neq []$ we proceed with the following derivation:

$$P \overset{a}{\rightarrowtail} \mathscr{P}, Q \overset{a}{\rightarrowtail} \mathscr{Q}, (\mathscr{P}|[L]|(\chi\, Q)) \neq [], ((\chi\, P)|[L]|\mathscr{Q}) \neq []$$

$\Rightarrow$ $\{$Def. $|[L]|\}$

$$P \overset{a}{\rightarrowtail} \mathscr{P}, Q \overset{a}{\rightarrowtail} \mathscr{Q}, \mathscr{P} \neq [], \mathscr{Q} \neq []$$

$\Rightarrow$ $\{$I.H.$\}$

$$(\mathsf{range}\, \mathscr{P}) = (\mathsf{range}\, \mathscr{Q}) = \{0, \iota\},$$

$\Rightarrow \quad \{\text{Def. } |[L]|\}$

$(\mathsf{range}\,(\mathscr{P}|[L]|(\chi\, Q))) = (\mathsf{range}\,((\chi\, P)|[L]|\mathscr{Q})) = \{0, \iota\}$

$\Rightarrow \quad \{\text{Def. } +^{\iota}\text{; Def. } \mathsf{range}\text{; Prop 8(i)}\}$

$\mathsf{range}((\mathscr{P}|[L]|(\chi\, Q))) + ((\chi\, P)|[L]|\mathscr{Q})) = \{0, \iota\}$

(ii) Suppose $\alpha = \surd$:

$P|[L]|Q \overset{\surd}{\rightarrowtail} (\mathscr{P}|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q})$

$\Rightarrow \quad \{\text{Def. } RTS \text{ semantics of } IML_k\}$

$P \overset{\surd}{\rightarrowtail} \mathscr{P}, Q \overset{\surd}{\rightarrowtail} \mathscr{Q}$

$\Rightarrow \quad \{\text{I.H.}\}$

$\iota \notin (\mathsf{range}\,\mathscr{P}), \iota \notin (\mathsf{range}\,\mathscr{Q})$

$\Rightarrow \quad \{\text{Def. } |[L]|\}$

$\iota \notin \mathsf{range}(\mathscr{P}|[L]|(\chi\, Q)), \iota \notin \mathsf{range}((\chi\, P)|[L]|\mathscr{Q})$

$\Rightarrow \quad \{\text{Def. } +^{\iota}\}$

$\iota \notin \mathsf{range}((\mathscr{P}|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q}))$

Case: $P|[L]|Q \overset{\alpha}{\rightarrowtail} \mathscr{P}|[L]|\mathscr{Q}$

In this case $\alpha = a \in \mathcal{A}_{IML_k}$, since $\alpha \in L \subseteq \mathcal{A}_{IML_k}$, and $\mathscr{P}|[L]|\mathscr{Q} \neq []$ by hypothesis:

$P|[L]|Q \overset{a}{\rightarrowtail} \mathscr{P}|[L]|\mathscr{Q} \neq []$

$\Rightarrow \quad \{\text{Def. } RTS \text{ semantics of } IML_k\text{; Def. } |[L]|\}$

$P \overset{a}{\rightarrowtail} \mathscr{P}, Q \overset{a}{\rightarrowtail} \mathscr{Q}, \mathscr{P} \neq [], \mathscr{Q} \neq []$

$\Rightarrow \quad \{\text{I.H.}\}$

$(\mathsf{range}\,\mathscr{P}) = (\mathsf{range}\,\mathscr{Q}) = \{0, \iota\}$

$\Rightarrow \quad \{\text{Def. } |[L]|\text{; } \mathscr{P}|[L]|\mathscr{Q} \neq [] \text{ by hypothesis.}\}$

$\mathsf{range}\,(\mathscr{P}|[L]|\mathscr{Q}) = \{0, \iota\}$

E.3 Proof of Theorem 7

Theorem 7. $\mathcal{R}_{IML_k}$ is total and functional.

Proof. $\mathcal{R}_{IML_k}$ *is total*: By induction on the structure, taking inaction and prefix as base cases, for which the assert is trivially proven. For the inductive step we show only the case $P\,|[L]|\,Q$ which is also very simple because $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ and $Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}$, for some $\mathscr{P}$ and $\mathscr{Q}$ by the I.H., hence, assuming $\alpha \notin L$, $P\,|[L]|\,Q \overset{\alpha}{\rightarrowtail} (\mathscr{P}|[L]|(\chi\,Q)) + ((\chi\,P)|[L]|\mathscr{Q})$ by the RTS semantics of IML_k (Fig. 6). The case $\alpha \in L$ is similar.

$\mathcal{R}_{IML_k}$ *is functional*: By induction on the length of the derivation for $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$. Let $n \geq 1$ be the length of the derivation for proving $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$.
Base case: Trivial since the only cases in which $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ can be derived with a proof of length 1 are those in which $\mathscr{P} = []$ or $\lambda.R \overset{\checkmark}{\rightarrowtail} \mathscr{P} = [R \mapsto \lambda]$, or $a.R \overset{a}{\rightarrowtail} \mathscr{P} = [R \mapsto \iota]$, for $a \in \mathcal{A}_{IML_k}$. In all these cases the assert easily follows from the definition of $[R \mapsto v]$.
Inductive step: The last assert of any proof of length $n > 1$ must be of the form $P + Q \overset{\alpha}{\rightarrowtail} \mathscr{P} + \mathscr{Q}$, or $X \overset{\alpha}{\rightarrowtail} \mathscr{P}$, or $P|[L]|Q \overset{\alpha}{\rightarrowtail} (\mathscr{P}|[L]|(\chi\,Q)) + ((\chi\,P)|[L]|\mathscr{Q})$, or $P|[L]|Q \overset{\alpha}{\rightarrowtail} \mathscr{P}|[L]|\mathscr{Q}$.

Case: $P + Q \overset{\alpha}{\rightarrowtail} \mathscr{P} + \mathscr{Q}$

$$P + Q \overset{\alpha}{\rightarrowtail} \mathscr{P}_1 + \mathscr{Q}_1, P + Q \overset{\alpha}{\rightarrowtail} \mathscr{P}_2 + \mathscr{Q}_2$$

$\Rightarrow$ {Def. RTS semantics of IML_k}

$$P \overset{\alpha}{\rightarrowtail} \mathscr{P}_1, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}_1, P \overset{\alpha}{\rightarrowtail} \mathscr{P}_2, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}_2$$

$\Rightarrow$ {I.H.}

$$\mathscr{P}_1 = \mathscr{P}_2, \mathscr{Q}_1 = \mathscr{Q}_2$$

$\Rightarrow$ {Algebra}

$$\mathscr{P}_1 + \mathscr{Q}_1 = \mathscr{P}_2 + \mathscr{Q}_2$$

Case: $X \overset{\alpha}{\rightarrowtail} \mathscr{P}$

$$X \overset{\alpha}{\rightarrowtail} \mathscr{P}_1, X \overset{\alpha}{\rightarrowtail} \mathscr{P}_2, X \triangleq P$$

$\Rightarrow$ {Def. RTS semantics of IML_k}

$$P \overset{\alpha}{\rightarrowtail} \mathscr{P}_1, P \overset{\alpha}{\rightarrowtail} \mathscr{P}_2$$

$\Rightarrow$ {I.H.}

$$\mathscr{P}_1 = \mathscr{P}_2$$

Case: $P|[L]|Q \overset{\alpha}{\rightarrowtail} (\mathscr{P}|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q})$

$$P|[L]|Q \overset{\alpha}{\rightarrowtail} (\mathscr{P}_1|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q}_1),$$
$$P|[L]|Q \overset{\alpha}{\rightarrowtail} (\mathscr{P}_2|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q}_2)$$

$\Rightarrow$ {Def. RTS semantics of IML_k}

$$P \overset{\alpha}{\rightarrowtail} \mathscr{P}_1, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}_1, P \overset{\alpha}{\rightarrowtail} \mathscr{P}_2, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}_2$$

$\Rightarrow$ {I.H.}

$$\mathscr{P}_1 = \mathscr{P}_2, \mathscr{Q}_1 = \mathscr{Q}_2$$

$\Rightarrow$ {Algebra}

$$(\mathscr{P}_1|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q}_1) = (\mathscr{P}_2|[L]|(\chi\, Q)) + ((\chi\, P)|[L]|\mathscr{Q}_2)$$

Case: $P|[L]|Q \overset{\alpha}{\rightarrowtail} \mathscr{P}|[L]|\mathscr{Q}$

$$P|[L]|Q \overset{\alpha}{\rightarrowtail} \mathscr{P}_1|[L]|\mathscr{Q}_1, P|[L]|Q \overset{\alpha}{\rightarrowtail} \mathscr{P}_2|[L]|\mathscr{Q}_2$$

$\Rightarrow$ {Def. RTS semantics of IML_k}

$$P \overset{\alpha}{\rightarrowtail} \mathscr{P}_1, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}_1, P \overset{\alpha}{\rightarrowtail} \mathscr{P}_2, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}_2$$

$\Rightarrow$ {I.H.}

$$\mathscr{P}_1 = \mathscr{P}_2, \mathscr{Q}_1 = \mathscr{Q}_2$$

$\Rightarrow$ {Algebra}

$$\mathscr{P}_1|[L]|\mathscr{Q}_1 = \mathscr{P}_2|[L]|\mathscr{Q}_2$$

E.4 Proof of Theorem 8

Theorem 8. For all $P, Q \in \mathcal{P}_{IML_k}$, $a \in \mathcal{A}_{IML_k}$, and unique functions $\mathscr{P}, \mathscr{P}' \in \Sigma_{\mathcal{P}_{IML_k}}$ such that $P \overset{a}{\rightarrowtail} \mathscr{P}$ and $P \overset{\checkmark}{\rightarrowtail} \mathscr{P}'$ the following holds: (i) $(\mathscr{P}\, Q) = \iota$ if and only if $P \xrightarrow{a} Q$; (ii) $(\mathscr{P}'\, Q) = \mathbf{rt}(P, Q)$. □

Proof. Proof of part (i). For the sake of conciseness, we prove both the direct (⇒) and the reverse (⇐) implication together. For the direct implication we proceed by induction on the length of the derivation for RTS semantics ($P \overset{a}{\rightarrowtail} \mathscr{P}$), while we use induction on the length of the derivation for the SOS ($P \xrightarrow{a} Q$) for the reverse implication. Let $n \geq 1$ be the length of the derivation for proving $P \overset{a}{\rightarrowtail} \mathscr{P}$ ($P \xrightarrow{a} Q$, respectively).

Base case: Trivial since the only case in which $P \overset{a}{\rightarrowtail} \mathscr{P}$ can be derived with a proof of length 1 and $(\mathscr{P}\,Q) = \iota$ is $a.Q \overset{a}{\rightarrowtail} \mathscr{P}$ with $\mathscr{P} = [Q \mapsto \iota]$. But $a.Q \overset{a}{\rightarrow} Q$ in the SOS definition of IML_k. On the other hand, the only case in which $P \overset{a}{\rightarrow} Q$ can be derived with a proof of length 1 is when $P = a.Q$, in which case $P \overset{a}{\rightarrowtail} [Q \mapsto \iota]$.
Inductive step: The last assert of any proof of length $n > 1$ must be of the form $P_1 + P_2 \overset{a}{\rightarrowtail} \mathscr{P}_1 + \mathscr{P}_2$, or $X \overset{a}{\rightarrowtail} \mathscr{P}_1$, or $P_1|[L]|P_2 \overset{a}{\rightarrowtail} (\mathscr{P}_1|[L]|(\chi\, P_2)) + ((\chi\, P_1)|[L]|\mathscr{P}_2)$, or $P_1|[L]|P_2 \overset{a}{\rightarrowtail} \mathscr{P}_1|[L]|\mathscr{P}_2$ and $P_1 + P_2 \overset{a}{\rightarrow} Q$, or $X \overset{a}{\rightarrow} Q$, or $P_1|[L]|P_2 \overset{a}{\rightarrow} Q$ (with $a \notin L$), or $P_1|[L]|P_2 \overset{a}{\rightarrow} Q$ (with $a \in L$), respectively.

Case: $\begin{cases} P_1 + P_2 \overset{a}{\rightarrowtail} \mathscr{P}_1 + \mathscr{P}_2, \text{ for } \Rightarrow \\ P_1 + P_2 \overset{a}{\rightarrow} Q, \text{ for } \Leftarrow \end{cases}$

$(\mathscr{P}_1 + \mathscr{P}_2)\,Q = \iota$

$\rightleftarrows$ {Def. RTS semantics of IML_k; Def. $(\mathscr{P}_1 + \mathscr{P}_2)$}

$P_1 \overset{a}{\rightarrowtail} \mathscr{P}_1, P_2 \overset{a}{\rightarrowtail} \mathscr{P}_2, (\mathscr{P}_1\,Q) = \iota$ or $(\mathscr{P}_2\,Q) = \iota$

$\Rightarrow$ {I.H.}
$\Leftarrow$ {I.H.; Unicity of $\mathscr{P}_1$, $\mathscr{P}_2$}

$P_1 \overset{a}{\rightarrow} Q$ or $P_2 \overset{a}{\rightarrow} Q$

$\rightleftarrows$ {Def. SOS of IML_k}

$P_1 + P_2 \overset{a}{\rightarrow} Q$

Case: $\begin{cases} X \overset{a}{\rightarrowtail} \mathscr{P}_1, X \overset{\Delta}{=} P_1, \text{ for } \Rightarrow \\ X \overset{a}{\rightarrow} Q, X \overset{\Delta}{=} P_1, \text{ for } \Leftarrow \end{cases}$

$(\mathscr{P}_1\,Q) = \iota$

$\Rightarrow$ {Def. RTS semantics of IML_k}
$\Leftarrow$ {Logics}

$P_1 \overset{a}{\rightarrowtail} \mathscr{P}_1, (\mathscr{P}_1\,Q) = \iota$

$\Rightarrow$ {I.H.}
$\Leftarrow$ {I.H.; Unicity of $\mathscr{P}_1$}

$P_1 \overset{a}{\rightarrow} Q$

$\overset{\Rightarrow}{\Leftarrow}$ {Def. SOS of IML_k}

$X \xrightarrow{a} Q$

$$Case: \begin{cases} P_1|[L]|P_2 \overset{a}{\rightarrowtail} (\mathscr{P}_1|[L]|(\chi\, P_2)) + ((\chi\, P_1)|[L]|\mathscr{P}_2), \text{ for } \Rightarrow \\ P_1|[L]|P_2 \xrightarrow{a} Q, a \notin L, \text{ for } \Leftarrow \end{cases}$$

$(\mathscr{P}_1|[L]|(\chi\, P_2)) + ((\chi\, P_1)|[L]|\mathscr{P}_2)\, Q = \iota$

$\Rightarrow$ {Def. RTS semantics of IML_k; Def. $+^\iota$; Def. $\cdot^\iota$}
$\Leftarrow$ {Def. $+^\iota$; Def. $\cdot^\iota$}

$P_1 \overset{a}{\rightarrowtail} \mathscr{P}_1, P_2 \overset{a}{\rightarrowtail} \mathscr{P}_2, a \notin L,$

$Q = Q_1|[L]|P_2$ for some Q_1 such that $(\mathscr{P}_1\, Q_1) = \iota$, or

$Q = P_1|[L]|Q_2$ for some Q_2 such that $(\mathscr{P}_2\, Q_2) = \iota$

$\Rightarrow$ {I.H.}
$\Leftarrow$ {I.H.; Unicity of $\mathscr{P}_1$, $\mathscr{P}_2$}

$Q = Q_1|[L]|P_2$ for some Q_1 such that $P_1 \xrightarrow{a} Q_1, a \notin L$ or

$Q = P_1|[L]|Q_2$ for some Q_2 such that $P_2 \xrightarrow{a} Q_2, a \notin L$

$\overset{\Rightarrow}{\Leftarrow}$ {Def. SOS of IML_k}

$P_1|[L]|P_2 \xrightarrow{a} Q, a \notin L$

$$Case: \begin{cases} P_1|[L]|P_2 \overset{a}{\rightarrowtail} \mathscr{P}_1|[L]|\mathscr{P}_2, \text{ for } \Rightarrow \\ P_1|[L]|P_2 \xrightarrow{a} Q, a \in L, \text{ for } \Leftarrow \end{cases}$$

$(\mathscr{P}_1|[L]|\mathscr{P}_2)\, Q = \iota$

$\Rightarrow$ {Def. RTS semantics of IML_k; Def. $\cdot^\iota$}
$\Leftarrow$ {Def. $\cdot^\iota$}

$P_1 \overset{a}{\rightarrowtail} \mathscr{P}_1, P_2 \overset{a}{\rightarrowtail} \mathscr{P}_2, a \in L,$

$Q = Q_1|[L]|Q_2$ for some Q_1, Q_2 such that$(\mathscr{P}_1\, Q_1) = \iota, (\mathscr{P}_2\, Q_2) = \iota$

$\Rightarrow$ {I.H.}

$\Leftarrow \quad \{\text{I.H.; Unicity of } \mathscr{P}_1,\ \mathscr{P}_2\}$

$Q = Q_1|[L]|Q_2$ for some Q_1, Q_2 such that$P_1 \xrightarrow{a} Q_1, P_2 \xrightarrow{a} Q_2, a \in L$

$\rightleftarrows \quad \{\text{Def. SOS of } IML_k\}$

$P_1|[L]|P_2 \xrightarrow{a} Q, a \in L$

Proof of part (ii). We proceed by induction on the length of the derivation for $P \overset{\surd}{\rightarrowtail} \mathscr{P}'$. Let $n \geq 1$ be the length of the derivation for proving $P \overset{a}{\rightarrowtail} \mathscr{P}'$.
Base case: Trivial since the only case in which $P \overset{\surd}{\rightarrowtail} \mathscr{P}'$ can be derived with a proof of length 1 and $(\mathscr{P}'\, Q) \neq 0$ is $\lambda.Q \overset{\surd}{\rightarrowtail} [Q \mapsto \lambda]$ and $\mathbf{rt}(\lambda.Q, Q) = \lambda$ by the SOS definition of IML_k and definition of $\mathbf{rt}$. In all other cases, $(\mathscr{P}'\, Q) = 0$ and there are no transitions $P \overset{\lambda}{\dashrightarrow} Q$, hence $\mathbf{rt}(P, Q) = 0$ by definition.
Inductive step: The last assert of any proof of length $n > 1$ must be of the form $P_1 + P_2 \overset{\surd}{\rightarrowtail} \mathscr{P}_1 + \mathscr{P}_2$, or $X \overset{\surd}{\rightarrowtail} \mathscr{P}_1$, or $P_1|[L]|P_2 \overset{\surd}{\rightarrowtail} (\mathscr{P}_1|[L]|(\chi\, P_2)) + ((\chi\, P_1)|[L]|\mathscr{P}_2)$.

Case: $P_1 + P_2 \overset{\surd}{\rightarrowtail} \mathscr{P}_1 + \mathscr{P}_2$

$(\mathscr{P}_1 + \mathscr{P}_2)\, Q$

$= \quad \{\text{Def. } (\mathscr{P}_1 + \mathscr{P}_2)\}$

$(\mathscr{P}_1\, Q) + (\mathscr{P}_2\, Q)$

$= \quad \{P_1 \overset{\surd}{\rightarrowtail} \mathscr{P}_1, P_2 \overset{\surd}{\rightarrowtail} \mathscr{P}_2; \text{ I.H.}\}$

$\mathbf{rt}(P_1, Q) + \mathbf{rt}(P_2, Q)$

$= \quad \{\text{SOS definition of } IML_k; \text{ Def. of } \mathbf{rt}\}$

$\mathbf{rt}(P_1 + P_2, Q)$

Case: $X \overset{\surd}{\rightarrowtail} \mathscr{P}_1, X \triangleq P_1$

$(\mathscr{P}_1\, Q)$

$= \quad \{P_1 \overset{\surd}{\rightarrowtail} \mathscr{P}_1; \text{ I.H.}\}$

$\mathbf{rt}(P_1, Q)$

$= \quad \{\text{SOS definition of } IML_k; \text{ Def. of } \mathbf{rt}\}$

$\mathbf{rt}(X, Q)$

Case: $P_1|[L]|P_2 \overset{\surd}{\rightarrowtail} (\mathscr{P}_1|[L]|(\chi\, P_2)) + ((\chi\, P_1)|[L]|\mathscr{P}_2)$

We observe that, from the *RTS* semantics of IML_k, if Q is neither of the form $Q_1|[L]|P_2$, nor of the form $P_1|[L]|Q_2$ then $((\mathscr{P}_1|[L]|(\chi\, P_2)) + ((\chi\, P_1)|[L]|\mathscr{P}_2))\, Q = 0$. On the other hand, we observe that the only $\dashrightarrow$ transitions allowed by the SOS definition of IML_k are to terms of the form $Q_1|[L]|P_2$ or $P_1|[L]|Q_2$, so, also $\mathbf{rt}(P_1|[L]|P_2, Q) = 0$ if Q is neither of the form $Q_1|[L]|P_2$, nor of the form $P_1|[L]|Q_2$. Let us assume, w.l.g., Q be of the form $Q_1|[L]|P_2$.

$$
\begin{array}{ll}
 & ((\mathscr{P}_1|[L]|(\chi\, P_2)) + ((\chi\, P_1)|[L]|\mathscr{P}_2))\, Q_1|[L]|P_2 \\
= & \quad \{\text{Def. } ((\mathscr{P}_1|[L]|(\chi\, P_2)) + ((\chi\, P_1)|[L]|\mathscr{P}_2))\} \\
 & ((\mathscr{P}_1|[L]|(\chi\, P_2))\, Q_1|[L]|P_2) + (((\chi\, P_1)|[L]|\mathscr{P}_2)\, Q_1|[L]|P_2) \\
= & \quad \{\text{Def. } ((\mathscr{P}_1|[L]|(\chi\, P_2)) + ((\chi\, P_1)|[L]|\mathscr{P}_2))\} \\
 & (\mathscr{P}_1\, Q_1) \\
= & \quad \{P_1 \overset{\surd}{\rightarrowtail} \mathscr{P}_1;\ \text{I.H.}\} \\
 & \mathbf{rt}(P_1, Q_1) \\
= & \quad \{\text{SOS definition of } IML_k;\ \text{Def. of } \mathbf{rt}\} \\
 & \mathbf{rt}(P_1|[L]|P_2, Q_1|[L]|P_2)
\end{array}
$$

The proof for the case in which Q is of the form $P_1|[L]|Q_2$ is similar.