ON A MATRIX ALGEBRA RELATED TO THE DISCRETE HARTLEY TRANSFORM

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Nota interna B4-53

Dicembre 1990
On a matrix algebra related to the discrete Hartley transform

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Abstract. A new matrix algebra \( \mathcal{H} \) including the set of real symmetric circulant matrices is introduced. It is proved that all the matrices of \( \mathcal{H} \) can be simultaneously diagonalized by the similarity transformation associated to the discrete Hartley transform. An application of this result to the solution of Toeplitz systems by means of the preconditioned conjugate gradient method is presented.

1. Introduction. Let \( S = (\sin \frac{2\pi i j}{n}) \), \( \tilde{S} = (\sqrt{\frac{2}{n+1}} \sin \frac{\pi(i+1)(j+1)}{n+1}) \) and \( C = (\cos \frac{2\pi i j}{n}) \), \( i, j = 0, \ldots, n - 1 \), be \( n \times n \) matrices, \( i \) the complex unit such that \( i^2 = -1 \) and \( A = \text{Circ}(a_0, \ldots, a_{n-1}) \) the circulant matrix whose first row is \( (a_0, \ldots, a_{n-1}) \), i.e. a matrix where the \((i, j)\)-entry is given by \( a_{k(i,j)} \), \( k(i, j) = j - i \mod n \). The structure of a circulant matrix is displayed below:

\[
A = \begin{pmatrix}
a_0 & a_1 & \cdots & a_{n-1} \\
a_{n-1} & a_0 & & \\
& \ddots & \ddots & \\
& & \ddots & a_1 \\
& & & a_1 & \cdots & a_{n-1} & a_0
\end{pmatrix}
\]

We denote by \( C_n \) the set of all the \( n \times n \) circulant matrices having entries in the complex field \( C \). The set \( C_n \) is an \( n \)-dimensional linear space, closed under the row by columns multiplication. That is \( C_n \) is a matrix algebra.

It is well known (see for instance [D]) that the Fourier matrix \( F = \frac{1}{\sqrt{n}}(C + iS) \), associated to the discrete Fourier transform (DFT) \( x \to Fx, x \in C^n \), is a unitary matrix that diagonalizes, by a similarity transformation, all the circulant matrices. In fact, the following relations hold:

\[
F^H F = FF^H = I, \\
F^H AF = \text{diag}(u_0, \ldots, u_{n-1}), \quad u = (u_i) = \sqrt{n}F^H Ae_1, \quad e_1 = (1, 0, \ldots, 0)^T, \quad (1.1)
\]

\( A \in C_n \).

Similar properties hold for the matrix \( \tilde{S} \) associated to the sine transform \( x \to \tilde{S}x \), and for the algebra \( T_n \) of matrices that can be expressed as a polynomial in \( Z = (z_{i,j}) \), \( z_{i,j} = 1 \) if \( |i - j| = 1 \), \( z_{i,j} = 0 \) otherwise [BC]:

\[
\tilde{S}S^T = \tilde{S}^2 = I, \\
\tilde{S}BS = \text{diag}(v_0, \ldots, v_{n-1}), \quad v_i = w_i/\theta_i, \quad (w_i) = \tilde{S}Be_1, \quad \theta_i = \sqrt{\frac{2}{n+1}} \sin \frac{\pi(i+1)}{n+1}, \quad (1.2)
\]

\( B \in T_n \).
Due to the computational efficiency of the well-known fast algorithms for the discrete Fourier transform and the sine transform computation [PFTV], equations (1.1) and (1.2) can be used for solving circulant and $T_n$ systems with a low computational cost (typically $O(n \log n)$ arithmetic operations). This fact has been used for devising fast algorithms for several matrix computations. For instance in [GS] circulant matrices are used in the solution of Toeplitz banded systems, in [BC], matrices in the class $T_n$ are used for the numerical computation of the eigenvalues of banded Toeplitz matrices. More recently (see [S], [CS], [C1], [C2], [C3]) circulant matrices have been used as preconditioners in the solution of Toeplitz systems by means of the conjugate gradient method. The result of [S], [CS], [C1] have been extended to the class $T_n$ improving the rate of convergence and reducing the cost per iteration [BDB]. The key fact, which these results are based on, is that, given a Toeplitz matrix $T$, the condition number of the matrix $S^{-1}T$ can be substantially reduced by choosing a suitable matrix $S$ in the class of real symmetric circulant matrices ([S] [CS], [C1]) or in the class $T_n$ ([BDB]).

In this paper, we introduce the Hartley matrix $H = \frac{1}{\sqrt{n}}(C + S)$ associated to the discrete Hartley transform (DHT) $x \rightarrow Hx$, (see [Br1], [Br2]), and define the algebra $\mathcal{H}_n$ of all the $n \times n$ real matrices that are simultaneously diagonalized by the similarity transformation associated to the matrix $H$ (we will prove that $H$ is orthogonal). We analyze the structure of $\mathcal{H}_n$ and we prove that this class includes the set of all the real symmetric circulant matrices. This fact allows us to determine more effective preconditioners of Toeplitz systems, which further reduce the condition number of the preconditioned matrix obtained by using circulant matrices. Actually, we extend to the class $\mathcal{H}_n$ all the results [CS], [S], [C1], proved in the case of circulant preconditioners and obtain algorithms having a better convergence, keeping the same computational cost per step. In fact, as it is easily verifiable (compare also [B], [SJBH] and proposition 2 in section 2), the discrete Hartley transform of real vector having $n$ components can be computed with a cost of $\frac{3}{2} n \log n$ additions and $n \log n$ multiplications (where we assume that $n$ is an integer power of 2, the cost bound is given up to additive constants and all the logarithms are to the base 2). The above cost bound, as we show in section 2, is the same cost bound of performing a DFT of a real vector. The equivalence of the computational costs of DFT and DHT has been pointed out also in [B], [SJBH].

2. The algebra $\mathcal{H}_n$. Let us start by analyzing some properties of the matrices $C$, $S$, $H$ defined in section 1.

Lemma 1. The following relations hold:

\[
\begin{align*}
C^2 + S^2 &= nI \\
CS &= SC = 0 \\
HH^T &= H^2 = I \\
JS &= SJ = -S \\
JC &= CJ = C \\
HJ &= JH
\end{align*}
\]
where $J$ is the permutation matrix

$$J = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots \\
0 & 1 & 0 & \cdots & 0
\end{pmatrix}.$$ 

**Proof.** From the relation $F^H F = F F^H = I$, we have $C^2 + S^2 = nI$, $CS = SC = 0$, therefore $H^T H = \frac{1}{n}(C^2 + S^2 + CS + SC) = I$. From the properties of trigonometric functions we obtain $SJ = JS = -S$, $JC = CJ = C$, whence $HJ = JH$. 

Let $\mathcal{A}_n$ and $\mathcal{B}_n$, denote the sets of real symmetric and skewsymmetric circulant $n \times n$ matrices, respectively; i.e.

- \{Circ(c_0, c_1, \ldots, c_{n-1}), \ c_i \in \mathbb{R}, \ i = 0, \ldots, n-1, \ c_i = c_{n-i}, \ i = 1, \ldots, \lfloor n/2 \rfloor\},
- \{Circ(0, c_1, \ldots, c_{n-1}), \ c_i \in \mathbb{R}, \ i = 1, \ldots, n-1, \ c_i = -c_{n-i}, \ i = 1, \ldots, \lfloor n/2 \rfloor\},

where $\mathbb{R}$ is the real field, and define the following class of $n \times n$ matrices

$$\mathcal{H}_n = \{E = A + JB : \ A \in \mathcal{A}_n, \ B \in \mathcal{B}_n\}.$$

Any matrix $E \in \mathcal{H}_n$ can be expressed as the sum of two independent matrices, the first being a symmetric circulant, the second a special Hankel matrix (a Hankel matrix has entries depending only on the sum of their subscripts), for instance, for $n = 5$

$$E = \begin{pmatrix}
a_0 & a_1 & a_2 & a_2 & a_1 \\
a_1 & a_0 & a_1 & a_2 & a_2 \\
a_2 & a_1 & a_0 & a_1 & a_2 \\
a_2 & a_2 & a_1 & a_0 & a_1 \\
a_1 & a_2 & a_2 & a_1 & a_0
\end{pmatrix} + \begin{pmatrix}
0 & b_1 & b_2 & -b_2 & -b_1 \\
b_1 & b_2 & -b_2 & -b_1 & 0 \\
b_2 & -b_2 & -b_1 & 0 & b_1 \\
-b_2 & -b_1 & 0 & b_1 & b_2 \\
-b_1 & 0 & b_1 & b_2 & -b_2
\end{pmatrix}.$$ 

Now we can prove the following results

**Proposition 1.** For any matrix $E \in \mathcal{H}_n$ the following relations hold

$$HEH = \text{diag}(d_0, \ldots, d_{n-1}),$$

$$(d_i) = \sqrt{n} H E e_1.$$

**Proof.** Let $A$ be any circulant matrix having real entries so that, by (1.1) $F^H A F = D_A$, where $D_A$ is a diagonal matrix; moreover

$$F^H A F = \frac{1}{n}(S A S + C A C) + i \frac{1}{n}(C A S - S A C),$$

whence

$$\text{Re}(D_A) = \frac{1}{n}(S A S + C A C) \quad \text{and} \quad \text{Im}(D_A) = \frac{1}{n}(C A S - S A C).$$
On the other hand

\[ HAH = \frac{1}{n}(C + S)(A + S) = \frac{1}{n}(CAC + SAS + CAS + SAC), \]

and, applying lemma 1 yields \( CAS + SAC = J(CAS - SAC) \), whence

\[ HAH = \text{Re}(DA) + J\text{Im}(DA). \]

For a general matrix \( E \in \mathcal{H}_n, E = A + JB, A \in \mathcal{A}_n, B \in \mathcal{B}_n, \) since \( F \) is a unitary matrix and \( A \) and \( B \) are real symmetric and real skewsymmetric matrices, respectively, we have that the diagonal matrices \( DA = F^HAF \) and \( DB = F^HBF \) have real and imaginary entries, respectively, i.e. \( \text{Im}(DA) = 0, \text{Re}(DB) = 0. \) This fact, together with lemma 1, implies that

\[ HEH = HAH + HJBH = HAH + JHBH = \text{Re}(DA) + \text{Im}(DB) = D. \]

The equation \((d_i) = \sqrt{n}HEe_i \) is obtained by applying both members of the equation \( HEH = \text{diag}(d_0, \ldots, d_{n-1}) \) to the vector \( e = (1, 1, \ldots, 1)^T \) since \( He = \sqrt{n}e_1. \)

Observe that, since \( \mathcal{H}_n = \mathcal{A}_n \oplus JB_n \) (where \( \oplus \) denotes the direct sum of subspaces and \( JB_n \) is the subspaces made up by the matrices \( JB, B \in \mathcal{B}_n \)), we have that \( \dim \mathcal{H}_n = \dim \mathcal{A}_n + \dim \mathcal{B}_n = n. \) In other words the \( n \)-dimensional algebra \( \mathcal{H}_n \) is isomorphic to the algebra of \( n \times n \) diagonal matrices over the real field.

The above theorem allows us devising methods to solve any system \( Ex = b, E \in \mathcal{H}_n, \) in \( O(n \log n) \) operations. This is due to the fact that for real vectors, the discrete Hartley transform can be computed with the same number of arithmetic operations in the real field, used in the computation of the discrete Fourier transform by means of FFT algorithms. We synthetize, in the following proposition, a straightforward extension of the radix-2 FFT algorithm.

**Proposition 2.** Let \( n \) be an integer power of 2. The components of the vector \( y = Hx, x, y \in \mathbb{R}^n, \) can be computed, given the components of the vector \( x \) and the entries of \( H, \) in at most \( \frac{3}{2} n \log_2 n \) additions and \( n \log_2 n \) multiplications of real numbers.

**Proof.** Let \( h_{ij}^{(n)} \) be the \((i,j)\)-entry of the \( n \times n \) Hartley matrix, and let \( n = 2m. \) Grouping together the odd components and the even components of \( x, \) respectively, yields

\[ y_i = \sum_{p=0}^{m-1} h_{i, 2p}^{(n)} x_{2p} + \sum_{p=0}^{m-1} h_{i, 2p+1}^{(n)} x_{2p+1}, \quad i = 0, \ldots, n - 1. \]

Moreover, since \( h_{i, 2p}^{(n)} = h_{i, p}^{(m)}, \) and \( h_{i, 2p+1}^{(n)} = \cos \frac{2\pi i}{n} h_{i, p}^{(m)} + \sin \frac{2\pi i}{n} h_{m-i, p}^{(m)}, \) where the subscript \( m - i \) has to be taken modulo \( m, \) we have that

\[ y_i = \sum_{p=0}^{m-1} h_{i, p}^{(m)} (x_{2p} + \cos \frac{2\pi i}{n} x_{2p+1}) + \sin \frac{2\pi i}{n} \sum_{p=0}^{m-1} h_{m-i, p}^{(m)} x_{2p+1}, \quad i = 0, \ldots, m - 1, \]

\[ y_{i+m} = \sum_{p=0}^{m-1} h_{i, p}^{(m)} (x_{2p} + \cos \frac{2\pi i}{n} x_{2p+1}) - \sin \frac{2\pi i}{n} \sum_{p=0}^{m-1} h_{m-i, p}^{(m)} x_{2p+1}, \quad i = 0, \ldots, m - 1, \]
That is, a Hartley transform of order \( n \) is reduced to two Hartley transforms of order \( n/2 \) and to performing at most \( n \) multiplications and \( \frac{3}{2}n \) additions. The result holds since the Hartley transform of order 1 does not require any operation.

We recall that the DFT of an \( n \) dimensional real vector can be obtained by computing a single DFT of an \( (n/2) \) dimensional complex vector [PFTV]; moreover, since the DFT of an \( m \) dimensional complex vector is performed in \( \frac{1}{2}m \log m \) complex multiplications and \( m \log m \) complex additions, the cost of computing the DFT of an \( n \) dimensional real vector amounts to \( \frac{1}{4}n \log n \) complex multiplications and \( \frac{1}{2}n \log n \) complex additions, that is, \( n \log n \) multiplications and \( \frac{3}{2}n \log n \) additions in the real field. This equivalence of the computational costs of DFT and DHT has been pointed out also in [B], [SJBH].

3. An application to preconditioning Toeplitz systems. Consider a linear system \( T_n x = b \), where \( T_n = (t_{i,j}) \) is a real symmetric \( n \times n \) Toeplitz matrix, that is, \( t_{i,j} = t_{i-j} \), \( i, j = 0, \ldots, n-1 \). Toeplitz matrices arise in many applications (see [Bunch] ) and concretely fast algorithms for the solution of these systems are extremely important.

The conjugate gradient method [GV] is particularly suitable for Toeplitz systems, in fact, the most expensive stages at each step of this algorithm consists in the computation of a matrix-by-vector product, where the matrix is Toeplitz. Due to this structure, such a product can be computed in \( O(n \log n) \) operations. However the algorithm takes \( n \) steps to arrive to completion so that the overall cost would be too high. A preconditioning strategy can be used for obtaining a good approximation to the solution in a constant number of steps. That is, a real symmetric positive definite matrix \( S_n \) (the preconditioner) should be determined so that

a) \( S_n \) is easily invertible (possibly in \( O(n \log n) \), arithmetic operations);

b) The matrix \( S_n \) is close to \( T_n \), that is, the eigenvalues of \( S_n^{-1}T_n \) are clustered around 1.

In this case the preconditioned conjugate gradient method computes an approximation to the solution in a number of steps which depends on the number of eigenvalues of \( S_n^{-1}T_n \) clustered around 1. The cost of each step is increased by the cost of solving a linear system with the matrix \( S_n \).

Condition a) is easily satisfied if we choose \( S_n \) in a matrix algebra made up by matrices which are simultaneously diagonalizable by a similarity transformation associated to a fast transform, for instance DFT, sine transform, or DHT. Condition b) has been proved for \( S_n \) belonging to the classes \( C_n \) and \( T_n \), [CS],[C1], [BDB]. In this section we prove that condition b) is satisfied by the matrix that solves the following minimum problem:

\[
\min_{S_n \in S_n} \|S_n - T_n\|_F, \quad (3.1)
\]

where \( \|X\|_F = \sqrt{\sum_{i,j} |x_{i,j}|^2} \) denotes the Frobenius norm. In [C1], [BDB] problem (3.1) has been treated in the cases where \( S_n = C_n \) and \( S_n = T_n \), respectively. In this section we deal with problem (3.1) in the case where \( S_n = H_n \). Moreover, since real symmetric circulant matrices are a subalgebra of \( H_n \), choosing \( S_n \) in the class \( H_n \) allows us to obtain a better approximation in the sense of (3.1).
Proposition 3. The matrix \( S_n = A_n + JB_n \in \mathcal{H}_n \) defined by
\[
\begin{align*}
  a_0 &= t_0, \\
  a_i &= \frac{(n-i)t_i + it_{n-i}}{n}, \quad b_i = \frac{t_i - it_{n-i}}{n}, \quad i = 1, \ldots, n - 1,
\end{align*}
\]
minimizes \( ||S_n - T_n||_F \) in the class \( \mathcal{H}_n \), where \( T_n = (t_{i,j}) \), \( t_{i,j} = t_{|i-j|} \).

Proof. The result is obtained by setting to zero the partial derivatives of \( ||S_n - T_n||_F \) with respect to \( a_0, a_i, b_i, i = 1, \ldots, n - 1 \).

In the following, under the same assumptions on the matrix \( T_n \) of [C1] and [BDB], we study the spectral properties of the matrix \( S_n^{-1}T_n \). More precisely we suppose that the matrices \( T_n, n = 1, 2, \ldots, \) are finite sections of a singly infinite symmetric matrix \( T_\infty \), generated by the real valued function \( f(z) = \sum_{j=-\infty}^{+\infty} t_j z^j \) defined on the unit circle in the complex plane and that \( f \) belongs to the Wiener class, that is, \( \sum_{j=-\infty}^{+\infty} |t_j| < +\infty \). The following result is an extension to the class \( \mathcal{H}_n \) of analogous results holding for \( C_n \) and \( T_n \).

Proposition 4. Let \( T_n \) be a Toeplitz matrix defined, as a finite section, from the function \( f(z) = \sum_{j=-\infty}^{+\infty} t_j z^j \) belonging to the Wiener class. Then

i) for any \( \epsilon > 0 \) there exists an integer \( n_0 \) such that for any \( n > n_0 \) the spectrum of \( S_n \) lies in the interval \([f_{\min} - \epsilon, f_{\max} + \epsilon]\), where \( f_{\min} \) and \( f_{\max} \) are the extremal values of \( f \) on the unit circle;

ii) the spectrum of \( S_n^{-1}T_n \) is clustered around 1, that is, for any \( \epsilon > 0 \) there exist integers \( k \) and \( n_0 \) such that for any \( n > n_0 \) the number of eigenvalues \( \lambda_i \) of \( S_n^{-1}T_n \) such that \( |\lambda_i - 1| < \epsilon \) is less than \( k \).

Proof. Concerning i), we observe that, if \( \lambda_j(S_n) \) denotes the \( j \)-th eigenvalue of \( S_n \), then
\[
\lambda_j(S_n) = a_0 + \frac{2}{n} \sum_{k=1}^{n-1} a_k ((n-k) \cos \frac{2\pi k(j-1)}{n} - \sin \frac{2\pi k(j-1)}{n}) =
\]
\[
\text{Re} \left( \sum_{k=-n+1}^{n-1} a_k e^{\frac{2\pi i k(j-1)}{n}} \right) - \frac{2}{n} \sum_{k=1}^{n-1} a_k k \cos \frac{2\pi k(j-1)}{n} - \frac{2}{n} \sum_{k=1}^{n-1} a_k \sin \frac{2\pi k(j-1)}{n}.
\]

Property i) follows since the sequences
\[
\left\{ \frac{2}{n} \sum_{k=1}^{n-1} a_k k \cos \frac{2\pi k(j-1)}{n} \right\} \quad \text{and} \quad \left\{ \frac{2}{n} \sum_{k=1}^{n-1} a_k \sin \frac{2\pi k(j-1)}{n} \right\}
\]
tend to zero. Concerning ii) it is sufficient to show that the eigenvalues of \( S_n^{-1}(T_n - S_n) \) are clustered around zero, in fact \( S_n^{-1}T_n = S_n^{-1}(T_n - S_n) + I \). Let \( \epsilon > 0 \) be fixed, since \( f \) belongs to the Wiener class we can choose \( N \) such that \( \sum_{j=N+1}^{+\infty} |t_j| < \epsilon \). The matrix \( T_n - S_n = T_n - A_n - JB_n \), can be split as
\[
W_n^{(N)} + Z_n^{(N)} + E_n^{(N)},
\]
where the first two matrices agree with the \((n-N) \times (n-N)\) leading principal submatrices of \(T_n - A_n\) and \(JB_n\), respectively. We observe that:

\[
\text{rank}(E_n^{(N)}) \leq 2N,
\]

\[
||W_n^{(N)}||_1 \leq \frac{2}{n} \sum_{k=1}^{n-N-1} k|t_{n-k} - t_k| \leq \frac{2}{n} \sum_{k=1}^{N} k|t_k| + 4 \sum_{k=N+1}^{n} |t_k|,
\]

\[
||Z_n^{(N)}||_1 \leq \frac{2}{n} \sum_{k=1}^{\left[\frac{n-1}{2}\right]} |t_k - t_{n-k}| \leq \frac{2}{n} \sum_{k=1}^{n-1} |t_k|,
\]

where \(||X||_1 = \max_j \sum_i |x_{i,j}|\) denotes the 1-norm. Now, let \(n_0 > N\) be such that \(\frac{1}{n_0} \sum_{k=1}^{N} k|t_k| < \epsilon\) and \(\frac{1}{n_0} \sum_{k=1}^{n_0-1} |t_k| < \epsilon\), then, for any \(n > n_0\) we have

\[
||W_n^{(N)} + Z_n^{(N)}||_1 \leq ||W_n^{(N)}||_1 + ||Z_n^{(N)}||_1 \leq 8\epsilon.
\]

Hence, by Cauchy interlace theorem [W], the eigenvalues of \(T_n - S_n\) are clustered around zero, except at most \(k = 2N\) of them. Applying Courant-Fischer minimax characterization [W] to the matrix \(S_n^{-1}(T_n - S_n)\), we have

\[
\lambda_i(S_n^{-1}(T_n - S_n)) < \frac{\lambda_i(T_n - S_n)}{f_{\text{min}}},
\]

for \(n\) sufficiently large, hence, even the spectrum of \(S_n^{-1}(T_n - S_n)\) is clustered around zero, that is for any \(\epsilon > 0\) there exist integers \(k\) and \(n_0\) such that for any \(n > n_0\) the number of eigenvalues \(\lambda_i\) of \(S_n^{-1}T_n\) which verify \(|\lambda_i - 1| > \epsilon\) is less than \(k\).

References.


