A NOTE ON THE QUADRANT INTERLOCKING FACTORIZATION

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ABSTRACT

The Quadrant Interlocking Factorization (QIF) proposed by D. J. Evans and several co-workers, is shown to be equivalent to 2x2 block LU factorization.

A permutation of rows and columns is presented, which transforms the matrices W and Z of the Quadrant Interlocking Factorization into two matrices L and U which are 2x2 block unit lower triangular and 2x2 block upper triangular, respectively. As a consequence the questions addressed on the QIF can be stated in terms of 2x2 block LU factorization.
1. INTRODUCTION

The Quadrant Interlocking Factorization (QIF) has been presented as an attractive alternative to LU factorization, well suitable for the implementation on Parallel Computers [1, 2, 3, 4, 5, 7, 9]; see also [6, pp. 173 and 194].

Let us consider a linear system \( Ax = b \), where \( A = (a_{ij}) \) is a real nonsingular \( nxn \) matrix. The main idea of the method is a factorization of \( A \) of the form

\[
A = WZ,
\]

with \( W = (w_{ij}) \) and \( Z = (z_{ij}) \) \( nxn \) matrices which have the structure

\[
W = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 1 & 0 & 0 & 0 & 0 & * \\
* & * & 1 & 0 & 0 & * & * \\
* & * & * & 1 & 0 & * & *
\end{pmatrix},
\]

\[
Z = \begin{pmatrix}
* & * & * & * & * & * & * \\
* & * & * & * & * & * & 0 \\
0 & * & * & * & * & 0 & 0 \\
0 & 0 & * & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & 0 & 0 \\
* & * & * & * & * & 0 & 0 
\end{pmatrix},
\]

where "*" denotes a generally nonzero entry.

The computation of the matrices \( W \) and \( Z \) is carried out as follows.

1) Let \( z_{i1} = a_{i1}, z_{i1} = a_{i1}, i=1, 2, \ldots, n \)

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2) Solve the n-2 2x2 linear systems

\[
\begin{align*}
\frac{w}{i1} + \frac{z}{i1} &= \frac{a}{i1}, \\
\frac{w}{i1} + \frac{z}{i1} &= \frac{a}{i1},
\end{align*}
\]

\(i=2,\ldots,n-1.

3) Update A as follows

\[
a_{ij} \leftarrow a_{ij} - \frac{w}{i1} \frac{z}{i1} - \frac{w}{i1} \frac{z}{i1}
\]

\(i=1,2,\ldots,n\)

\(j=1,2,\ldots,n\)

In this way the first and last columns (rows) of \(W\) (2) have been computed. To complete the factorization, the same process is applied to the inner quadrant of the updated version of \(A\).

**EXAMPLE 1.1**

Let \(A\) be the 5x5 matrix

\[
\begin{bmatrix}
1 & 2 & 3 & 2 & 2 \\
6 & 16 & 16 & 11 & 6 \\
9 & 32 & 53 & 32 & 9 \\
6 & 11 & 16 & 10 & 6 \\
2 & 2 & 3 & 2 & 1
\end{bmatrix}
\]

then \(A = W \cdot Z\), where

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 2 \\
3 & 4 & 1 & 4 & 3 \\
2 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & 2 & 2 \\
0 & 2 & 4 & 3 & 0 \\
0 & 6 & 3 & 0 & 0 \\
0 & 3 & 4 & 2 & 0 \\
2 & 2 & 3 & 2 & 1
\end{bmatrix}
\]

2. MAIN RESULT

Let us consider the permutation matrix \(P\) corresponding to the permutation \((1, n, 2, n-1, \ldots)\), namely

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\( p = (p_{ij}) \), \( p_{ij} = \begin{cases} 1 & \text{if } i \text{ odd, } j = (i+1)/2, \\ 1 & \text{if } i \text{ even, } j = n - i/2 + 1, \\ 0 & \text{otherwise.} \end{cases} \)

Given two matrices \( W \) and \( Z \) with the structure shown in the previous section, one can readily verify that

\[
\begin{pmatrix}
I & 0 & \cdots & 0 \\
\$ & I & \cdots & \$ \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
\end{pmatrix}
\]

is upper triangular, if \( n \) is odd.

\[
\begin{pmatrix}
I & 0 & \cdots & 0 \\
\$ & I & \cdots & \$ \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
\end{pmatrix}
\]

where \( I \) denotes the \( 2 \times 2 \) identity matrix, and \( "$\) denotes a generally nonzero \( 2 \times 2 \) block.

Analogously, it is easy to see that the matrix \( W Z P^T \) is \( 2 \times 2 \) block upper triangular.

The linear system \( A \mathbf{x} = \mathbf{b} \), can be written \( W Z \mathbf{x} = \mathbf{b} \),

from which \( PWZP^T \mathbf{x} = \mathbf{b} \),

and \( PWZP^T \mathbf{x} = \mathbf{b} \),

i.e. \( L U \mathbf{y} = \mathbf{b} \), with \( \mathbf{y} = \mathbf{P} \mathbf{x} \).

Thus the decomposition \( A = WZ \) is equivalent to an \( L U \)
decomposition where $L$ is 2x2 block unit lower triangular, and $U$ is 2x2 block upper triangular.

**Example 2.1**

Let $A$, $W$, $Z$ be the matrices of example 1.1. Let $P$ be the permutation matrix

$$
\begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
$$

then $L = P W P^T$, $U = P Z P^T$, where

$$
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 2 & 2 & 2 \\
0 & 2 & 0 & 1 \\
0 & 3 & 4 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 2 & 2 \\
2 & 1 & 2 & 2 \\
0 & 0 & 3 & 2 \\
0 & 0 & 0 & 3 \\
\end{array}
$$

and $FAF = 6 6 10 11 16$

3. CONCLUSIONS

As a consequence, it turns out that the questions, addressed by Evans and colleagues, concerning: conditions on the existence of QIF without pivoting, numerical stability of the method, parallel implementation, block extensions of QIF, rate of convergence of SOR methods based on QIF, can be stated in terms of block LU factorizations.
Moreover it is easy to verify that the modification of QIF presented in [2], after the same permutation, becomes equivalent to the classical LU factorization, as the authors themselves note in [7].
REFERENCES


