Summary: A Koehler-type method to obtain lower bounds for the eigenvalues of a certain class of operators is presented. The general properties required of a problem for the technique to work are discussed and the connection with other classical methods is analyzed.

1. Introduction

In this paper I elaborate a method of obtaining lower bounds for the eigenvalues of an operator $A = A' + B$, when the spectral properties of $A'$ are known and $B$ meets suitable conditions of complete continuity. The present work makes precise, and expands to some extent, a comment of results obtained in [1] in the study of the infinitesimal stability of homogeneously deformed plates and shells.

Although the method has been worked out independently, it comes close to a technique proposed by KOEHLER to estimate the eigenvalues of infinite matrices [2]. (1) My approach is different from KOEHLER's, however, and more advantageous in several respects. Firstly, a perhaps minor condition that is present in KOEHLER's paper is not required here. Secondly, the approach emphasizes the role of those general characters on which the classification of a problem is based; this makes it easier to discuss the range of validity of the present method and the strategies that make it applicable.

(1) I am indebted to Professor W. VELTE for making me aware of this paper and for many useful discussions on the subject.
When combined with upper bounds, the use of lower bounds supplies a direct measure of the error affecting numerical estimates of eigenvalues. The practical importance of techniques that yield lower bounds has long been recognized; an extended account is found in [3].

A distinguishing feature common to all methods of obtaining lower bounds is that some set of basic information and/or a priori estimates about the problem is needed. For the method I propose such a set is described in Section 2; this method provides arbitrarily close lower bounds to the eigenvalues of \( A \) in terms of their Rayleigh-Ritz approximations. An analysis of the role of the hypotheses, together with a brief discussion of some classes of problems that meet them, is presented in Sections 3 and 4. In particular, I point out connections with the techniques based on the construction of intermediate problems (see \([4],[5],[6] \) and \([7]\) that form the most important class of methods for obtaining lower bounds for eigenvalues. I examine in some detail the possibility of relaxing the basic hypotheses while maintaining the effectiveness of the present technique. In this respect, some attention is given to a comparison with the truncated operator method proposed by WEINBERGER [3] and recently revisited by FOX and STADTER [8].

Finally, in Section 5 I present a numerical application. As noted in Sections 3 and 4, sometimes suitable strategies can bring problems (that do not fit in with the hypotheses of Section 2) into the range of applicability of KOEHLER's technique. These strategies do in general require slight modifications of the method outlined in the previous sections. Rather than testing the numerical accuracy of the technique, I use this example to give an account of the main differences and also to indicate how to overcome the major numerical difficulties.
2. Description of the method

Consider the following eigenvalue problem:

\[ Au = \lambda u \quad \text{for} \quad u \in \mathcal{D}(A), \]

where \( A \) is a lower bounded self-adjoint operator defined on a dense linear manifold \( \mathcal{D} \) of a Hilbert space \( H \), with scalar product \((\cdot, \cdot)\). Assume that \( A \) admits a decomposition

\[ A = A' + B, \tag{2.1} \]

where \( A' \) is self-adjoint on \( \mathcal{D} \), \( B \) is defined over \( \mathcal{D} \), and both forms \((B^*B, \cdot, \cdot)\) and \((\cdot, \cdot)\) are completely continuous with respect to \((A', \cdot, \cdot)\), \( B^* \) being the adjoint of \( B \). In particular, the last assumption implies that for any \( \varepsilon > 0 \) there is a number \( \varepsilon(\varepsilon) \) such that:

\[ (B^*Bu, u) \leq \varepsilon (A'u, u) + \varepsilon(\varepsilon) (u, u) \tag{2.2} \]

for all \( u \in \mathcal{D} \).

In order to find lower bounds for the eigenvalues of \( A \), we require knowledge of:

a) The spectral properties of \( A' \);

b) A pair of numbers \( \delta \in [0, 1) \) and \( \kappa \) with the property

\[ |(Bu, u)| \leq \delta (A'u, u) + \kappa (u, u) \tag{2.3} \]

for all \( u \in \mathcal{D} \). \(^{(2)}\)

\(^{(2)}\) The existence of such a pair is a consequence of (2).
Under the above circumstances $A'$ has an unbounded discrete spectrum $\lambda_1' \leq \lambda_2' \leq \lambda_3' \leq \ldots$, and any corresponding family of eigenfunctions $\{u_1', u_2', u_3', \ldots\}$ is a basis of $\mathcal{H}$. By assumption, both the spectrum and a complete family of eigenfunctions are known.

In the following, $\mathcal{U}_n$ is defined to be

$$\mathcal{U}_n = \text{span} \{u_1', u_2', \ldots, u_n'\},$$

and $\mathcal{U}_n^\perp$ stands for its orthogonal complement in $\mathcal{H}$. Furthermore, I shall denote by

$$\beta_n : \mathcal{H} \mapsto \mathcal{U}_n$$

the orthogonal projection onto $\mathcal{U}_n$, and by

$$\mathcal{B}(1) = \left\{ u \in \mathcal{H} : (u, u) = 1 \right\}$$

the unit ball in $\mathcal{H}$.

The method is founded on the lemma:

**Lemma**—Let $\mathcal{S}_n$ and $\beta_n^2$ be defined by

$$\mathcal{S}_n := \lambda_n' \left(1 - \mathcal{S}_n'\right) - \mathcal{K},$$

$$\beta_n^2 := \sup_{v \in \mathcal{U}_n \cap \mathcal{B}(1)} (B(v, w))^2,$$

where $v \in \mathcal{U}_n \cap \mathcal{B}(1)$ and $w \in \mathcal{U}_n^\perp \cap \mathcal{B}(1)$. Then, $A$ is positive if and only if $\exists \ n$ such that

$$\min_{\mathcal{U}_n \cap \mathcal{B}(1)} (A(v, v)) > \frac{\beta_n^2}{\mathcal{S}_n} \geq 0.$$
Proof.

Put \( v := \varphi_n u \) and \( w := (1 - \varphi_n)u \) for \( u \in \mathcal{D} \). Then, from the orthogonality of \( \mathcal{U}_n \) and \( \mathcal{U}_n^\perp \) with respect to \( A' \) and from (4), it follows that

\[
(Au, u) = (Av, v) + 2(Bv, w) + (Aw, w) \geq (Av, v) - 2\beta_n \|v\| \|w\| + (Aw, w).
\]

Hence, the inequalities \((Aw, w) \geq g_n(w, w)\) (from (3) and (4)) and

\[
-2\beta_n \|v\| \|w\| > -2\beta_n^2 (v, v) - \frac{1}{\gamma^2} (w, w)
\]

(which holds for any \( \gamma > 0 \)) together yield

\[
(Au, u) \geq \left[ \min_{\mathcal{U}_n \cap \mathcal{Y}_n} (Av, v) - \frac{\gamma^2}{2\beta_n^2} (v, v) + \left[ g_n - \frac{1}{\gamma^2} \right] (w, w) \right].
\]

Inequalities (5) and (6) imply that \( A \) is positive, since we can always find a number \( \gamma \) that makes the square brackets in (6) positive.

To prove the necessity of (5), observe that since

\[
\lim_{n \to \infty} g_n = +\infty
\]

the inequality (5) is always satisfied for \( n \) large enough.

Moreover, by the Cauchy-Schwartz inequality and (2), one obtains from (4) that

\[
\frac{\beta_n^2}{g_n} \leq \frac{(B^* Bv, v)}{g_n} \leq \frac{1}{(1-\delta) - \frac{\varepsilon}{1-\delta}} \left[ \frac{\varepsilon}{k_n} \frac{k_n'}{k_n + 1} + \frac{\varepsilon}{k_n + 1} \right]
\]

for any \( \varepsilon \) and \( n \). Thus, as \( \varepsilon \) may be chosen as small as we wish and \( k_n' \to \infty \), it follows that

\[
\lim_{n \to \infty} \frac{\beta_n^2}{g_n} = 0.
\]
On the other hand, if $A$ is positive

$$\min_{U_n \cap \bar{B}(1)} (Av, v) \geq \lambda_1 > 0,$$

$\lambda_1$ being the smallest eigenvalue of $A$. Thus (5) are satisfied for $n$ large enough.

As $U_n$ is finite-dimensional, the form $(Bv, w)$ is weakly continuous in $U_n \times U_n^\perp$ with respect to the norm induced by the inner product of $H$. If $(Bv, w)$ is extended to $U_n \times U_n^\perp$ by continuity, the extended form attains both its maximum and minimum on $U_n \cap \bar{B}(1) \times U_n^\perp \cap \bar{B}(1)$. It follows that

$$\beta_n^2 = \max_{U_n \cap \bar{B}(1)} \left\{ \max_{U_n^\perp \cap \bar{B}(1)} (Bv, w)^2 \right\},$$

and, by the projection theorem,

$$\beta_n^2 = \max_{U_n \cap \bar{B}(1)} \left\{ \|Bv\|^2 - \|nBv\|^2 \right\}.$$ 

Then, checking the inequalities (5) requires finite algebra only, and the lemma furnishes a basis for an expedient computational procedure delivering lower bounds for the eigenvalues of $A$.

Remark 1 - Notice that conditions (5) remain necessary and sufficient even if one uses for $\gamma_n$ any positive quantity such that $(Aw, w) \geq \gamma_n (w, w)$ for all $w \in U_n^\perp$ and, for $\beta_n^2$, any greater value than the one defined in (4), provided that $\frac{\beta_n^2}{\gamma_n} \to 0$ when $n \to \infty$. Reference to this is made at several points in the following; in particular, the use of quantities greater than (9) for $\beta_n^2$ may be convenient in practice.

Set $A_\gamma = A + \gamma I$, where $I$ denotes the identity in $H$ and $\gamma$ any real number. Then,
(2.10) \( (A_g u, u) > 0 \) for \( \forall u \in \mathcal{D} \setminus \{0\} \iff -\gamma < \min \limits_{\mathcal{D} \cap \mathcal{B}(1)} (Au, u) = \mu_1. \)

By (7), for any fixed \( \gamma \), \( (\varepsilon_n + \gamma) \) is positive for \( n \) large enough; moreover, \( (A_g w, w) \geq (\varepsilon_n + \gamma)(w, w) \) for all \( w \in \mathcal{U}_n \). From remark 1 it follows that \( A_g \) is positive if and only if for some \( n \) the inequalities

\[
(2.11) \quad \min \limits_{\mathcal{U}_n \cap \mathcal{B}(1)} (Av, v) + \gamma > \frac{\beta_n^2}{\varepsilon_n + \gamma} \geq 0
\]

are satisfied.

For any fixed \( \gamma \), (11) may be regarded as conditions on \( \gamma \); thus from (10) and (11) we obtain

\[
(2.12) \quad -\gamma^{(n)} := \frac{1}{2} \left[ (\mu_1^{(n)} + \varepsilon_n) - \sqrt{(\mu_1^{(n)} - \varepsilon_n)^2 + 4\beta_n^2} \right] \leq \mu_1
\]

where \( \mu_1^{(n)} := \min \limits_{\mathcal{U}_n \cap \mathcal{B}(1)} (Av, v) \) is the Rayleigh-Ritz approximation of \( \mu_1 \). Moreover, since for any \( \gamma \) such that \( -\gamma < \mu_1 \) the inequalities (11) are satisfied for some \( n \), it follows that quantity \( \mu_1 + \gamma^{(n)} \) can be made arbitrarily small for suitable choices of \( n \).

Lower bounds for eigenvalues of order greater than one are obtained similarly.

Let \( k \) and \( n \), \( n \geq k \), be chosen, and denote by \( \mu_1^{(n)} \) and \( u_i^{(n)} \) \( (i=1,2,\ldots,n) \) the Rayleigh-Ritz approximations of order \( n \) for the first \( n \) eigenvalues and eigenfunctions, respectively. If one considers the restriction of \( (A_g \cdot, \cdot) \) to the subspace

\[
\mathcal{U}_k^{(n)} := \{ u \in \mathcal{D} : (u, u_i^{(n)}) = 0 \quad i=1,2,\ldots,k-1 \}
\]

(10) becomes

\[
(2.13) \quad (A_g u, u) > 0 \quad \forall u \in \mathcal{U}_k^{(n)} \setminus \{0\} \iff -\gamma < \min \limits_{\mathcal{U}_k^{(n)} \cap \mathcal{B}(1)} (Au, u). \]
Now, $\mathcal{V}_k^{(n)} = U_{n,k} \oplus U_n^\perp$, with $U_{n,k} := \text{span} \{ u_k^{(n)}, \ldots, u_n^{(n)} \} \subset U_n$.

Therefore, the same argument used in the lemma shows that $A_{\Psi}$ is positive in $\mathcal{V}_k^{(n)}$ if

$$\min_{U_{n,k} \cap \mathcal{B}(1)} (Av, v) + \Psi > \frac{\beta_{n,k}^2}{\varepsilon_n + \Psi} \geq 0$$

with $\beta_{n,k}^2$ defined by

$$\beta_{n,k}^2 := \sup_{v \in U_{n,k} \cap \mathcal{B}(1)} (Bv, w)^2$$

and $w \in U_n^\perp \cap \mathcal{B}(1)$.

Observing that $\beta_{n,k}^2 \leq \beta_{n,2}^2$ and recalling remark 1, $A_{\Psi}$ is positive if $\Psi$ satisfies (13) with $\beta_{n,k}^2$ replaced by $\beta_{n,2}^2$. Then, exploiting (13) and (14) one obtains

$$\frac{1}{2} \left[ (\mu_k^{(n)} + \varepsilon_n) - \sqrt{(\mu_k^{(n)} - \varepsilon_n)^2 + 4 \beta_{n}^2} \right]$$

where the latter inequality follows from the min-max principle, since the deficiency of $\mathcal{V}_k^{(n)}$ in $\mathcal{H}$ is $k-1$.

Thus, (15) generalizes (12) to eigenvalues of all orders. Convergence of the estimates (15) to $\mu_k$ is a consequence of the inequalities

$$-\mathcal{F}_k^{(n)} - \mu_k \leq \mu_k^{(n)}$$

and the property

$$\lim_{n \to \infty} \mu_k^{(n)} + \mathcal{F}_k^{(n)} = 0$$

(2) This follows from the inclusion $U_{n,k} \subset U_n$. 

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*(Note: The document is a mathematical text discussing the properties of eigenvalues and their estimates. The text contains mathematical expressions and inequalities, which are essential for understanding the context and the results presented. The text is extracted and formatted in a readable manner to ensure clarity and accessibility for the reader.)*
which is easily proved by use of (7) and (8). This incidently also establishes the convergence to \( \mu_k \) of the Rayleigh-Ritz approximations.

**Remark 2** By (12) and (13), the lemma has an easy corollary: \( \mu_k \geq 0 \) if and only if \( \exists n \) such that \( \mu_k^{(n)} \frac{\lambda_n}{\gamma_n} \geq 0 \).

This corollary can be used for an alternative proof of convergence to the higher order eigenvalues.

3. Comments and remarkable applications

Formulae (2.12) and (2.15) are similar to those given by KOEHLER in [2]. As a matter of fact, KOEHLER considered the reciprocal of the Rayleigh quotient and, consequently, was interested in upper bounds rather than in lower ones; the difference is immaterial for \( A \) positive, as KOEHLER explicitly assumed in his proof. In contrast, my above argument does not require such an assumption and shows that formulae (2.12) and (2.15) provide lower bounds in any case.

The approach makes evident a certain set of general properties of a problem that both provide the a priori estimates needed in KOEHLER's technique, and also guarantee the validity of relations (2.7) and (2.8) that are of the essence for convergence. In particular, the basic properties listed in Section 2 serve as a guide in recognizing cases to which the method applies.

The intermediate problems in both WEINSTEIN and ARONZAJIN's methods, [4] and [5], offer an example of cases falling within the range of applicability of the present technique; more generally, the same is true for almost all techniques based on the construction of comparison operators (vid. [6] and [7]). In fact, although all these envisage different types of problems and pursue different ideas, they generally
propose the study of finite-rank perturbations $B$ of a known operator $A$. Accordingly, $B^*B$ is completely continuous and formulae (2.12) and (2.15) can provide arbitrarily close lower bounds to the eigenvalues of the comparison operator under study. This may be an alternative to the use of more cumbersome algorithms.

An important class of problems where the required properties of $A'$ and $B^*B$ are recognizable by using familiar results in analysis occurs when $A$ is an elliptic differential operator of order $2m$, defined on a domain $\Omega \subset \mathbb{R}^n$. Quite commonly $B$ is then the function space $L^2(\Omega)$, and $\mathcal{D}$ is a linear manifold in $H^{2m}(\Omega)$. In this case, if $A'$ contains the principal part of $A$, one can use the equivalence between norms $\|\cdot\|_{2m}$ and $\|A'\| + \|\cdot\|$, and a classical interpolation theorem, (cf. [9], theorems 11.11 and 3.3), to show that $B^*B$ is completely continuous with respect to $A'$ if the order of $B$ is less than $m$.

Such a condition, for instance, is satisfied in the eigenvalue problems for Sturm-Liouville operators (3) that can be given the form

$$
(3.1) \quad -\frac{d^2u}{dx^2} + Bu = Cu,
$$

where $x$ belongs to some bounded interval $(a,b); u \in H^2(a,b); C$ is positive; and both $B$ and $C$ are operators of order 0. This is the general case for single equations (under suitable regularity of the coefficients) and for systems with constant coefficients. The operator on the left-hand side of (1) meets

(3) Estimates for the eigenvalues of Sturm-Liouville problems are obtainable in many ways; an effective and neat method, using the construction of intermediate problems again, has been proposed by WEINSTEIN [10].
the conditions of Section 2; thus, with the few changes required by the presence of the operator $C$, and illustrated by the numerical example of Section 5, convergent lower bounds to the eigenvalues can be obtained by the method outlined above.

4. Non-convergent estimates. Comparison with Weinberger's method

In the lemma, (2.7) and (2.8) are required only for proving the necessity of conditions (2.5) and for establishing the convergence of the method. If one gives up convergence, the assumptions yielding (2.7) and (2.8) can be dropped and the method applied to a larger set of problems.

Let $\mathcal{P} \subset \mathcal{H}$ and $\mathcal{Q} > 0$ be given in such a way that

\begin{equation}
(Au,u) \geq \mathcal{Q} (u,u) \quad \text{for all } u \in \mathcal{P}^\perp \oplus \mathcal{Q},
\end{equation}

where $\mathcal{P}$ is some subspace of finite dimension $n$. If one decomposes $\mathcal{H}$ according to $\mathcal{H} = \mathcal{P} \oplus \mathcal{P}^\perp$ and proceeds as in the first part of the lemma, inequality (2.6) still follows with $\mathcal{Q}$ in place of $\beta_n$ and $\beta_n^2$ replaced by

\begin{equation}
\beta^2 := \sup \{ (Av,w)^2 \} = \max \{ ||Av||^2 - ||QAV||^2 \},
\end{equation}

where $v \in \mathcal{P}^\perp \cap \mathcal{Q}(1)$ and $w \in \mathcal{P} \cap \mathcal{Q}(1)$, while $\Omega : \mathcal{H} \rightarrow \mathcal{P}$ is the orthogonal projection on $\mathcal{P}$. Hence, $A$ is positive if conditions (2.5), with $\mathcal{U}_n$ and $\mathcal{S}_n$ replaced by $\mathcal{P}$ and $\mathcal{Q}$, are satisfied; it follows that formulae (2.12) and (2.15) still yield lower bounds to the first $n$ eigenvalues of $A$.

Thus, lower bounds of KOEHLER-type are deducible whenever a pair $(\mathcal{P}, \mathcal{Q})$ with property (1) is known: no preliminary decomposition of $A$, nor the knowledge of any special basis for $\mathcal{H}$, is really needed. If one takes this point
of view, the a priori information required is the same as for the truncated operator method proposed by WEINBERGER [3]; in a sense, this is the minimum information needed in any technique delivering lower bounds in eigenvalue problems.

Other connections with WEINBERGER's method are also interesting.

To give a full account of WEINBERGER's method would be too long; here are sketched just few points following a restatement of the method due to FOX and STADTER [8]. To that paper the reader is sent back for more details and for the meaning of symbols not encountered so far. For brevity, attention is confined to the case envisaged in Sections 1.2–1.4 of [8], i.e., when \( \mathcal{A}_2 = \mathcal{A} \cap Q^L \) contains the null vector only.

According to FOX and STADTER, WEINBERGER's method requires the preliminary knowledge of a pair \((\mathcal{A}, \mathcal{S})\) with property (1) and makes use of a finite-dimensional subspace \(\mathcal{Q}\) arbitrarily chosen in \(\mathcal{Q}\). The method consists in the study of the spectral problem for the truncated operator (cf. [8], formula (2.3))

\[
A_2 := A - (1-Q)^T (A-I^T) (1-Q),
\]

where \(Q\) is a suitable linear map from \(\mathcal{A}\) into \(\mathcal{Q}\). As \(A_2 \leq A\), the eigenvalues of \(A_2\) bound from below the corresponding eigenvalues of \(A\). FOX and STADTER show that, with the choice suggested by WEINBERGER, the finite-dimensional subspace \(m_2 = \mathcal{A} \cap Q^\perp\) (4) is invariant under \(A_2\), and that \(A_2\) equals \(g_1\) on \(m_2^\perp\); hence, the exact eigenvalues of \(A_2\) can be found by finite algebra confining attention to \(m_2\).

If one chooses \(Q = \mathcal{A}\), \(Q\) becomes the orthogonal pro-

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(4) The symbol \(\otimes\) denotes the linear span.
jection of \( \mathcal{H} \) onto \( \mathcal{B} \), and \( A_2 \) comes to coincide with \( A \) over \( \mathcal{B} \).

From the lemma in Section 2, then, \( A_2 \) is positive if the inequalities (2.5) are satisfied, with \( \mathcal{B} \) and \( g \) in place of \( U_n \) and \( g_n \), respectively. It follows that the estimates derived from (2.5) by use of (2.12) and (2.15) are in fact lower bounds for the eigenvalues of \( A_2 \) and, in particular, that WEINBERGER's method is expected to provide better lower bounds for \( A \) as it brings to the exact eigenvalues of \( A_2 \). WEINBERGER's method is more cumbersome than KOEHLER's when applied in its general form, but, in the simple case \( \mathcal{Q} = \mathcal{B} \), it is probably equivalent to that from a computational point of view.

In both methods \( g \) is the best lower bound obtainable on the basis of (1) for the eigenvalues \( \mu_k \geq g \). For WEINBERGER's method this is so because \( g \) is an eigenvalue of infinite multiplicity for \( A_2 \); in the present case, this follows from the definition of \( \nu_k^{(n)} \), cf. (2.15), and the inequality:

\[
(4.4) \quad \frac{1}{2} \left[ (\mu_k^{(n)} + g) - \sqrt{(\mu_k^{(n)} - g)^2 + 4\beta^2} \right] \leq g
\]

for all \( k \leq n \). On the other hand, it is easily seen that \( \nu_k^{(n)} \) converges to \( \mu_k \), for all \( \mu_k < g \), when in KOEHLER's technique a sequence of subspaces \( \beta_1 < \beta_1 < \beta_2 < \ldots \) is used such that the corresponding \( \beta_1 \)'s tend to 0. In this case, WEINBERGER's operators also yield convergent estimates. Unfortunately, an explicit characterization of sequences with this property is lacking.

As a concluding remark, observe that the information contained in (1) is available when the known part of \( A \) is dominant, at least asymptotically. In elliptic problems, for instance, this occurs if \( A' \) contains the principal part of \( A \). In that case, it is rather easy to obtain perhaps crude inequalities like (1), and use the technique outlined above.
5. A numerical application

There are cases where the conditions of Section 2, or Section 4, do not occur openly but require some manipulation of the eigenvalue problem. It happens, for instance, with the elliptic problems in one-variable when one tries to reduce them to form (3.1). As a consequence of these manipulations a positive operator appears on the right side of the eigenvalue problem, and the present technique requires slight changes in order to cope with this. The example discussed in the present section expounds the main differences involved.

Consider the eigenvalue problem

\[
- \frac{d^2 u}{dx^2} = \mu (1 - \sin x) u \quad \text{with} \quad u(0) = u(\pi) = 0.
\]

To take account of the factor \((1 + \sin x)\) in (1), observe that, from the min-max property, \(\lambda < \mu_k\) if the operator

\[
A(\lambda) = - \frac{d^2 u}{dx^2} - \lambda (1 + \sin x) u
\]

is positive on some subspace of deficiency \(k-1\) in \(\mathcal{D} = H^2(0,\pi) \cap H^1_0(0,\pi)\). Now, as \(\sin kx\) and \(k^2\), with integer \(k\), respectively, are the eigenfunctions and the eigenvalues of \(-\frac{d^2 u}{dx^2}\) under the boundary conditions of problem (1), and as \(A(\lambda)\) falls within the class of operators studied in Section 2, positivity of \(A(\lambda)\) is characterized by the lemma therein. Then, if one chooses the normalized eigenfunctions \(u_k = \sqrt{\frac{2}{\pi}} \sin kx\) as coordinate functions and denotes by \(u_1^{(n)}, u_2^{(n)}, \ldots, u_n^{(n)}\) and \(\mu_k^{(n)}\), all depending on \(\lambda\), the Rayleigh-Ritz approximations of order \(n\) for the eigenfunctions and the eigenvalues of \(A(\lambda)\), \(A(\lambda)\) is positive in \(\mathcal{V}_k = \{u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)}\}^\perp\) if the following inequalities hold true:

\[
\mu_k^{(n)}(\lambda) > \frac{\lambda^2 \beta_n^2}{q_n(\lambda)} \geq 0.
\]
According to (4.1) and (4.2), in (2) \( q_n(\lambda) \) and \( \beta_n^2 \) are chosen to be
\[
q_n(\lambda) = (n+1)^2 - 2|\lambda|,
\]
(5.3)
\[
\beta_n^2 = \sup((1+\sin x)v, w)^2,
\]
where \( v \in \mathcal{L}_n \cap \mathcal{P}(t) \) and \( w \in \mathcal{L}_n \cap \mathcal{J}(t) \). Hence, \( \lambda < \mu_k \) if it satisfies (2); the lower upper bound \( \lambda_k^{(\mu)} \) of such \( \lambda \)'s are then the best lower approximation of \( \mu_k \) obtainable from (2).

Numbers \( \mu_k^{(n)}(\lambda) \) are the eigenvalues of the nxn matrix \( A^{(n)}(\lambda) := [(A_n\delta_{ul} u_{1l})] \), whose entries are
\[
A^{(n)}(\lambda) = \begin{cases} 0 & \text{if (k+1) is odd}, \\ \frac{k^2}{(k-1)^2} - \lambda(\delta_{kl}) + \frac{2}{n} (\frac{1}{(k+1)^2 - 1} - \frac{1}{(k-1)^2 - 1}) & \text{if (k+1) is even}, \end{cases}
\]
where \( \delta_{kl} \) are the Kronecker symbols. As is easily checked, for \( n \geq 3 \) it is possible to make explicit the dependence of \( \mu_k^{(n)} \) on \( \lambda \) and calculate \( \lambda_k^{(n)} \) directly. Recalling remark 1, and making use in (2) of the inequality \( \beta_n^2 \leq 1/4 \), computation of \( \lambda_k^{(n)} \) yields:
\[
\lambda_1^{(1)} = 0.52807 , \quad \lambda_1^{(2)} = 0.53598 , \quad \lambda_2^{(2)} = 2.16930 , \quad \lambda_1^{(3)} = 0.53770 , \quad \lambda_2^{(3)} = 2.25158 , \quad \lambda_3^{(3)} = 4.89997 .
\]

Problem (1) was considered also by COLLATZ (cf. [11], p. 187) and WEINSTEIN [10]. For a comparison, COLLATZ's approximation to \( \mu_1 \) was 0.53880, and WEINSTEIN's were 0.53940 and 2.35775, respectively, for \( \mu_1 \) and \( \mu_2 \). On the other hand, for \( \mu_1 \) and \( \mu_2 \) Rayleigh-Ritz upper bounds are 0.54088 and 2.38228.

For \( n \geq 3 \) it is not possible to find the explicit dependence of \( \mu_k^{(n)} \) on \( \lambda \), therefore one has to resort to numerical trials, or iterative methods, in order to approach
\( \lambda^{(n)}_k \). This is the main difference from the cases envisaged in Section 2.

For fixed \( n \) and \( k \), successive approximations to \( \lambda^{(n)}_k \) can be calculated using in (2) the linearized expression for \( \mu^{(n)}_k (\lambda) \):

\[
\mu^{(n)}_k (\lambda_o) + \mu^{(n)}_k (\lambda_o) (\lambda - \lambda_o) \quad (5)
\]
evaluated at some known \( \lambda_o \); at which the inequalities (2) hold true. Then, the above procedure can be repeated starting from the new value of \( \lambda \) so calculated, or from an intermediate value between this and \( \lambda_o \), according to whether or not (2) are satisfied.

To find out \( \mu^{(n)}_k (\lambda_o) \), observe that \( \mu^{(n)}_k (\lambda) \) is an eigenvalue of the algebraic problem

\[
(5.5)
\]

\[
(\xi^{(n)}(\lambda) \cup_k (\lambda) = \mu^{(n)}_k (\lambda) \cup_k (\lambda) \quad (5.6)
\]

where the \( \mathbb{R}^n \)-vector \( \xi^{(n)}(\lambda) \) consists of the components of the unit vector \( u^{(n)}_k (\lambda) \) with respect to the basis \( u_1, u_2, \ldots, u_n \). Therefore, differentiating (6) at \( \lambda_o \) yields

\[
(5.7)
\]

As \( \cup_k (\lambda_o) \perp \text{Range}\{\xi^{(n)}(\lambda_o) - \mu^{(n)}_k (\lambda_o) 1\} \), from (7) one obtains \( \mu^{(n)}_k (\lambda_o) \) by multiplying both terms in (6) by \( \cup_k (\lambda_o) \). Using equivalent notations, this turns out to be

\[
\mu^{(n)}_k (\lambda_o) = \left( \xi(\lambda_o) u^{(n)}_k (\lambda_o), u^{(n)}_k (\lambda_o) \right),
\]

an expression that delivers \( \mu^{(n)}_k (\lambda_o) \) in terms of \( u^{(n)}_k \) evaluated at \( \lambda_o \).

(5) Dot stands for derivation with respect to \( \lambda \).
REFERENCES


Note for the printer

(Correspondence between symbols in the manuscript and characters to use in print)

$\mathcal{D}$ ....... capital german $D$

$\mathcal{H}$ ....... " " $H$

$\mathcal{U}$ ....... " " $U$

$\mathcal{B}$ ....... " " $B$

$\mathcal{V}$ ....... " " $V$

$p$ ....... " " $P$

$q$ ....... " " $Q$

$u, v, w$ ........ small german $u, v, w$.

$\mathcal{U}$ ........ light italic $u$.

$\alpha, \beta, \gamma, \delta, \epsilon, \eta, \kappa, \lambda, \mu, \pi, \rho$ ........ greek letters

$A, B, C, Q$ ........ capital english letters

$\mathcal{A}$ ........ light italic letters