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Abstract

The theory of variational inequalities is applied to a problem of lubrication. A method for solving numerically the relevant variational inequality is presented.
1. Introduction

The problem of studying the hydrodynamic lubrication of a complete Journal bearing of finite length, open to both ends at the atmospheric pressure can be stated as follows:

Problem A (1)

Let \( Q = \left\{ (\theta, y) \in \mathbb{R}^2; -\infty < \theta < \infty, |y| < b \}, b > 0 \). To find a function \( v(\theta, y) \) such that

\[
(1.1) \quad L v = -\frac{\partial}{\partial \theta} \left( h^3 \frac{\partial v}{\partial \theta} \right) - \frac{\partial}{\partial y} \left( h^3 \frac{\partial v}{\partial \theta} \right) = \varepsilon \sin \theta, \text{ with }
\]

\[
(1.2) \quad h = 1 + \varepsilon \cos \theta \quad \text{ and } \quad 0 < \varepsilon < 1,
\]

\[
(1.3) \quad v(\theta, b) = v(\theta, -b) = 0, \quad -\infty < \theta < \infty
\]

\[
(1.4) \quad v(\theta, y) = v(\theta + 2\pi, y), \quad (\theta, y) \in \bar{Q}.
\]

It is easy to see [4], [13] that the solution of Problem A exists, is unique and is negative when \( \pi + 2k\pi < \theta < 2\pi + 2k\pi, k = 0, \pm 1, \pm 2, \ldots \). Yet, under normal condition of operation, negative pressure cannot be supported by the lubricating fluid; hence equation (1.1) makes sense only where \( v > 0 \) and everywhere in \( \bar{Q} \) it must be \( v > 0 \). Where the film yields to slight substmospheric pressure a bubble full of air (i.e. a region of cavitation) is formed whose shape is "a priori" unknown. Hence we may state the following free boundary problem [4], [3]

\[ (1) \] A detailed description of the physical background of this problem can be found e.g. in [2], [9].
Problem B

To find a function \( p \) defined in \( \overline{Q} \) and an open set \( \Omega \) such that

\[
(1.5) \quad p(0, b) = p(0, -b) = 0, \quad -\infty < b < \infty
\]

\[
(1.6) \quad p(\theta, y) = p(\theta + 2\pi, y), \quad (\theta, y) \in \overline{Q}
\]

\[
(1.7) \quad p > 0 \text{ and } \int_{\Omega} p = \varepsilon \sin \theta \text{ in } \Omega
\]

\[
(1.8) \quad p = 0, \quad \frac{dp}{dn} = 0 \text{ on } \partial \Omega \cap Q.
\]

\( \frac{dp}{dn} \) denotes differentiation with respect to the outward pointing normal to \( \partial \Omega \cap Q \). We note that (1.8) is meaningful only when \( \partial \Omega \cap Q \) is a regular curve.

The free boundary Problem B can be studied with the techniques described by H. Lewy and G. Stampacchia in [7]. This has been noticed first in [8] and subsequently investigated in [4], [10], [1].

We shall denote by \( C^1(\overline{Q}) \) the space of functions which are continuously differentiable in \( \overline{Q} \) and periodic in \( \theta \) with period \( 2\pi \). The functions of \( C^1_{\infty}(\overline{Q}) \) which vanish near \( \partial Q \) shall be called \( C^1_{\infty}(Q) \). Moreover, the completion of \( C^1_{\infty}(\overline{Q})(C^1_{\infty}(Q)) \) with respect to the norm

\[
||v||_{H^1} = \left( \int_{\Omega} |v|^2 d\theta dy + \int_{\Omega} |\nabla v|^2 d\theta dy \right)^{1/2}, \quad \Omega = \{ (\theta, y); 0 < \theta < 2\pi, |y| < b \},
\]

will be denoted by \( H^1_{\infty} \), \( (H^1_{\infty})_0 \).

Let us consider the following closed convex subset of \( H^1_{\infty} \)

\[ IK = \{ v \in H^1_{\infty}, \ v > 0 \text{ in } Q \}, \text{ then Problem B is expressed by the variational inequality} \]

\[
(1.9) \quad p \in IK, \quad \int_{\Omega} h^2 \nabla p \cdot \nabla (v-p) d\theta dy \geq \varepsilon \int_{\Omega} \sin \theta (v-p) d\theta dy, \quad \forall v \in IK.
\]
Theorem 1.1

There exists one and only one solution $p \in C^1(\Omega)$ of variational inequality (1.9).

Proof

Uniqueness and existence follow from the well-known results of [11], whereas the periodicity conditions make our problem slightly different from those considered in [7]. In order to overcome the difficulty and obtain the $C^1$-regularity, we transform the strip $Q$ in the annulus

$$0 = \{ x \in \mathbb{R}^2, a < |x| < a + 2b \}, \quad x = (x_1, x_2) \quad |x| = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad a > 0$$

with the transformation

$$x_1 = (a+b+y) \cos \theta, \quad x_2 = (a+b+y) \sin \theta.$$  

Under (1.10) the variational inequality (1.9) becomes

$$(1.11) \quad p \in IK, \quad \int h^2 \sum_{i,j=1}^{2} a_{ij} (v-p) x_j \, dx \geq \int \frac{x_2^2}{2} (v-p) \, dx, \quad \forall v \in IK,$$

with

$$a_{11} = \frac{x_1^2}{|x|^2} + \frac{x_2^2}{|x|^2}, \quad a_{12} = a_{21} = \frac{x_1 x_2}{|x|^2} - \frac{x_1^2}{|x|^2} - \frac{x_2^2}{|x|^2}, \quad a_{22} = \frac{x_2^2}{|x|^2} + \frac{x_1^2}{|x|^2},$$

$$h = \frac{1}{|x|} \left( 1 + \frac{x_1^3}{|x|^3} \right) \quad \text{and} \quad IK = \{ v \in H^1_0(0), \quad v \geq 0 \ \text{in} \ \Omega \}.$$

The results of regularity of [7] are directly applicable to (1.11) and we obtain $p \in C^1(\Omega).$  \hfill \Box

Remark 1.1

By Theorem 1.1, the solution $p$ of (1.9) belongs to $C^1(\bar{\Omega}) \cap H^1_{\infty,0}$, hence $p$ satisfies (1.5), (1.6). Moreover, if we define
\( m = \{ (0, y) \in Q; p > 0 \}, \) and take in (1.9) variations localized in \( \Omega, \) it is easy to prove also (1.7). The conditions of transition (1.8) would require a more detailed analysis of the regularity of the free boundary (see e.g. [4]).

2. Numerical Solution

Our numerical treatment of the variational inequality (1.9) is based on the method described in [12] and [6].

Let us consider the sequence of functions

\[
\delta_m(t) = \begin{cases} 
1, & \text{for } t < -\frac{1}{m} \\
-mt, & \text{for } -\frac{1}{m} \leq t < 0 \\
0, & \text{for } t \geq 0 
\end{cases}
\]

and the non-linear Dirichlet problem

\[(2.1) \quad p_m \in H^1_0(\Omega), \quad Lp_m = \max(0, -\frac{x_2}{|x|} \delta_m(p_m) + \frac{x_2}{|x|} \text{ in } \Omega).\]

By theorems 2, 3 and 4 of [7] the sequence \( \{p_m\} \) is non-decreasing and converges uniformly together with the first partial derivatives to the solution \( p \) of the variational inequality (1.11); moreover we have

\[ |p_m - p| \leq \frac{1}{m}. \]

Problem 2.1 can be restated in the rectangle \( R \) as follows

\[(2.2) \quad P_m = p_m - u \in H^1_{\Omega}, \quad LP_m = F_m(\theta, P_m) \text{ in } R,\]

where \( F_m(\theta, P_m) = \phi(\theta) \delta_m(P_m + u), \phi(\theta) = \max(0, -\sin \theta), \) and \( u \) is solution of

\[ u \in H^1_{\Omega}, \quad \iint_{R} h^3u \cdot \nabla v \, d\theta d\gamma = \epsilon \iint_{R} \sin v \, d\theta d\gamma \quad \forall \nu \in H^1_{\Omega}. \]
In order to solve problem (2.2), let us consider, for every integer \( m \), the iterative procedure introduced in [12],

\[
(2.3) \quad z(s+1) \in H^1_{x_0}, \quad Lz(s+1) + k \Phi(\theta) z(s+1) = F_m(\theta, z^s) + k \Phi(\theta) z^s,
\]

where \( k \) is a positive constant such that \( k > m \), and let us denote by \( \{ z^s \} \) the sequence obtained from (2.3) starting with \( z^0 = 0 \).

By theorem 5 of [12] it follows that the sequence \( \{ z^s \} \) is non-decreasing and converges uniformly with its first derivatives to the solution of problem (2.2) for every \( m \). We note that all the functions \( z^s \) are of class \( C^2 \).

To derive a finite difference approximation to (2.2)-(2.3) we make use of the integration method described in [14]. Let us choose a rectangular mesh with sides parallel to the \( \theta, y \) axes, and mesh spacings \( \Delta \theta = \frac{2\pi}{N} \), \( \Delta y = \frac{2b}{M+1} \) (\( N,M \) integers).

We take into account the condition of periodicity considering as coincident the sides \( \theta = 0 \) and \( \theta = 2\pi \) of the rectangle \( R \) and treating their mesh points as internal points. If we number the mesh points \( P_i, i=1,2,\ldots,M+N \) from left to right, top to bottom we can write the systems of linear equations obtained from our difference scheme as follows

\[
(2.2') \quad A \hat{W} = F(W)
\]

\[
(2.3') \quad (A+k\Psi) v(s+1) = F(v^s) + k\Psi v^s,
\]

where \( v^s \) is the vector whose components are \( v^s_i = v^s(P_i) \), \( F(\hat{z}) \) is the column vector with components \( F(P_i, z(P_i)) \), and \( \Psi \) is the diagonal matrix whose elements are \( \Phi(P_i) \).

The matrix \( A \) is block tridiagonal of the form

\[
A = \begin{bmatrix}
C & D & & \\
D & C & D & \\
& & \ddots & \\
& & & : & D & C & D \\
& & & : & & : & D & C
\end{bmatrix}
\]
where $D$ is a diagonal matrix of order $N$ and $C$ is a tridiagonal matrix of order $N$. More precisely, if we put

$$
\theta_i = (i-1) \Delta \theta, \quad i=1, \ldots, N, \quad y_j = -b + j \Delta y, \quad j=1, \ldots, M
$$

$$
\theta_{i+\frac{1}{2}} = \theta_i + \frac{\Delta \theta}{2},
$$

then we have for the elements of the matrix $C$

$$
C_{i,i} = \frac{h(\theta_{i+\frac{1}{2}}) + h(\theta_{i-\frac{1}{2}})}{\Delta \theta^2} + \frac{h(\theta_i)}{\Delta y^2}, \quad i=1, \ldots, N,
$$

$$
C_{i,i+1} = C_{i+1,i} = -\frac{h(\theta_{i+\frac{1}{2}})}{\Delta \theta^2}, \quad i=1, \ldots, N-1
$$

$$
C_{1,N} = C_{N,1} = -\frac{h(\theta_{N-\frac{1}{2}})}{\Delta \theta^2},
$$

and for those of the matrix $D$

$$
D_{i,i} = \frac{h(\theta_i)}{\Delta y^2}, \quad i=1, \ldots, N.
$$

The matrix $A$ has positive diagonal elements and non-positive off-diagonal elements. Moreover, $A$ is irreducibly diagonally dominant and symmetric; hence [14] $A$ is an $M$-matrix. From theorem 3.4 of [6] follows that $\lim_{s \to \infty} V_s = W$.

In exactly the same way as in [6] we can also show that the solutions of (2.2')-(2.3') converge to the solutions of (2.2)-(2.3); moreover all the considerations of [6] concerning the estimation of the error, remain true.
Results of the numerical experiments

We exhibit in Figure 1 the pressure distributions which we have obtained applying the method described in this paper for different values of the parameters $\varepsilon$ and $b$. These "level curves" have been obtained directly by the computer.

References


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