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CENTRO STUDI CALCOLATRICI ELETTRONICHE

del C. N. R.

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Introduction

A cylindrical bearing is considered (radius $R$, width $b$, radial clearance $c$), which is referred to a system of cylindrical coordinates $(l, \rho, \phi, z)$ with the $z$-axis along the axis of symmetry and $O$ in the central plane. The anomaly $\rho$ is counted in that plane from an axis directed as the load acting on the journal; a first, fixed frame $T$ is thus defined.

A second, moving frame $T'(O, x, y, z)$ is thought as attached to the journal, with the $x$-axis as axis of symmetry of the journal and $O$ in the central plane of the journal. Hence, the position of the journal is completely known once the eccentricity $\epsilon = |\Omega|$, the anomaly $\beta$ of $\Omega$, and the Euler angles $\theta, \psi, \phi$ of $T'$ with respect to $T$ are assigned.

Then, if $\eta$ is the constant lubricant viscosity, $\omega$ the steady rotational speed of the journal in radians per unit time, and $h$ the thickness of the film

$$h(\rho, \phi) = c - \epsilon \cos (\rho - \beta) + \theta^2 \sin (\rho - \psi),$$

Reynolds' equation for the pressure $p$ within the lubricant in a short bearing can be written as follows:

$$\frac{\partial}{\partial \rho} \left[ \frac{h^3 \partial p}{6\eta \partial \rho} \right] = -2\epsilon \cos (\rho - \beta) + (\omega - 2\beta) \epsilon \sin (\rho - \beta)
+ 2\beta \psi \sin (\rho - \psi) + (\omega - 2\psi) \epsilon \cos (\rho - \psi).$$

This equation determines a unique function $p(\rho, \phi)$ when the boundary conditions

$$p \left( \rho, \pm \frac{b}{2} \right) = 0$$

are also required.

Though $p$ [as a solution of (2)] is thus defined over all the field $0 \leq \rho \leq 2\epsilon, -\frac{b}{2} \leq z \leq \frac{b}{2},$ it has physical significance only where it is positive. Within the region where $p$ is positive

Nomenclature

- $a$ = eccentricity ratio $a = c/e$
- $A$ = value of $a$ in the steady state
- $a_1$ = perturbation of $a$ from $A$
- $b$ = width of bearing
- $c$ = radial clearance of bearing
- $C_1, C_2$ = moments of inertia of rigid journal
- $e$ = eccentricity of journal center: $e = |\Omega|$
- $F$ = force due to the lubricant

and acting on the journal

$M_n, M_\alpha = \text{components of } M \text{ along the vector } \Omega \text{ and in a direction normal to } \Omega \text{ on the journal's central plane}$

$M_1, M_2 = \text{components of } M \text{ along the nodal line } l \text{ and in a direction normal to } l \text{ on the journal's central plane}$

$O = \text{journal's center}$

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where \( a_i, \beta_i \) are quantities of the first order, together with \( \theta \) and their derivatives toward \( \tau = \omega \). The equation to be satisfied by \( p \) becomes

\[
\frac{\partial}{\partial \xi} \left[ \frac{A^2}{6 \eta_0} \frac{\partial (p - p_i)}{\partial \xi} \right] = -2a_i' \cos \alpha' \\
+ 2A \beta_i' A \cos \alpha - a_i \sin \alpha - 2 \beta_i' \sin (\alpha + \beta - \psi) \\
- (1 - 2 \beta_i') A \cos (\alpha + \beta - \psi),
\]

where a prime denotes a derivative toward \( \tau \).

At the same time, the expression of \( h \) can be split into the sum of the steady-state value \( h_0 \) and a small addenda of the first order

\[
h = h_0 + [a_i \cos \alpha - \beta_i' \sin (\alpha + \beta - \psi)].
\]

Solution of Reynolds Equation

By integration of equation (5), with the specification (6) for \( h \)
and the usual boundary conditions for \( \xi = \frac{b}{2} \), one obtains for \( p \)
the expression

\[
p = p_i + \frac{\eta \omega}{c(1 + A \cos \alpha)} \left( \frac{b^2}{4} - \frac{1}{\xi} \right) \left\{ \begin{array}{l}
2 \theta' \sin (\alpha + \beta - \psi) \\
+ (1 - 2 \theta') \cos (\alpha + \beta - \psi)
\end{array} \right\} \\
+ \frac{6}{1 + A \cos \alpha} \sin \alpha (\alpha + \beta - \psi) \cdot \frac{\xi}{\xi} \\
- 6a_i' \cos \alpha - 3(2 \beta_i' \alpha - a_i) \sin \alpha
\]

Under the circumstances envisaged here, the region filled by the lubricant (i.e., the region where (7) assigns a positive value to \( p \)) is specified by

\[\alpha_i(\xi) \leq \alpha \leq \alpha_i(\xi) + \pi\]

where

\[\alpha_i(\xi) = \frac{1}{A} \left[ 2 \xi a_i' - \frac{1}{3 \xi} (2 \theta' \sin (\beta - \psi)) \right] \]

\[+ (1 - 2 \beta') \cos (\beta - \psi) \]

Note, in fact, that \( p(\alpha_i(\xi), \xi) \) and \( p(\alpha_i(\xi) + \pi, \xi) \) are zero to within quantities of the second order.

Naturally, to calculate resultant force and moment of the couple acting on the journal because of the pressure field, we need only integrate over \( (0, \pi) \) with respect to \( \alpha \); the integrals over \((0, a_i)\) and \((\pi, \pi + \alpha_i)\) are quantities of the second order. The components of the force \( F_a \) along the vector \( \Omega \), and in a direction normal to it, are given by

\[
F_a = \int_{-b/2}^{b/2} d\xi \int_{0}^{\pi} p \sin \alpha R \, d\alpha,
\]

\[
F_a = \int_{-b/2}^{b/2} d\xi \int_{0}^{\pi} p \cos \alpha R \, d\alpha
\]

at the same time the components of the vector \( \mathbf{M} \) representing the moment of the couple along the same directions are

\[
M_a = -\int_{-b/2}^{b/2} \xi d\xi \int_{0}^{\pi} p \sin \alpha R \, d\alpha,
\]

\[
M_a = \int_{-b/2}^{b/2} \xi d\xi \int_{0}^{\pi} p \cos \alpha R \, d\alpha.
\]

Formulas (8) lead to the specification of \( F_a, F_n \), already given by Holmes [1]

\[
F_a = \eta \beta \omega \left( \begin{array}{l}
\pi \\
\frac{A}{4 (1 - A^2)^{3/2}} (1 - 2 \beta) + \frac{2A}{(1 - A^2)^{3/2}} \sin \beta
\end{array} \right) \\
+ \frac{\pi}{4 (1 - A^2)^{3/2}} \sin \beta
\]

\[
F_n = \eta \beta \omega \left( \begin{array}{l}
\pi \\
\frac{A}{4 (1 - A^2)^{3/2}} (1 - 2 \beta) + \frac{2A}{(1 - A^2)^{3/2}} \\
+ \frac{\pi}{4 (1 - A^2)^{3/2}}
\end{array} \right)
\]

This is so because, within the limits of small perturbations, the rocking of the journal does not influence the resultant force. Change in this force from the steady-state value is related to only the parameters which describe the "parallel" component of journal motion.

The rocking of the journal gives rise instead to a couple, which is conversely not influenced by parallel whirl

\[M_a = \frac{\eta \beta \omega}{120 \omega_1^3} \left( \begin{array}{l}
\pi \\
\frac{1 + 3A^2}{2 (1 - A^2)^{3/2}} + \frac{\pi}{(1 - A^2)^{3/2}} \sin \beta
\end{array} \right)
\]

\[+ \frac{4A}{(1 - A^2)^{3/2}} \sin \beta - \frac{6}{(1 - A^2)^{3/2}} \cos \beta
\]

\[+ \frac{\pi}{(1 - A^2)^{3/2}} \cos \beta
\]

\[M_n = \frac{\eta \beta \omega}{120 \omega_1^3} \left( \begin{array}{l}
\frac{2A(3 + 5A^2)}{(1 - A^2)^{3/2}} + \frac{\pi}{(1 - A^2)^{3/2}} \sin \beta
\end{array} \right)
\]

\[+ \frac{\pi}{(1 - A^2)^{3/2}} \sin \beta - \frac{6}{(1 - A^2)^{3/2}} \cos \beta
\]

\[+ \frac{\pi}{(1 - A^2)^{3/2}} \cos \beta
\]

\[= \frac{A}{(1 - A^2)^{3/2}} \cos \beta
\]

Notice that, even during a conical whirl of small amplitude, the precession angle \( \psi \) and its derivative need not, of course, be small.

\[1 \text{ Numbers in brackets designate References at end of paper.}
\]

**Nomenclature**

- \( p \) = pressure within the lubricant
- \( p_i \) = steady-state value of \( p \)
- \( R \) = radius of bearing
- \( T(O, x, y, z) \) = moving Cartesian frame attached to the journal
- \( T(O, \rho, r, \xi) \) = fixed cylindrical frame attached to the bearing
- \( t \) = time
- \( \alpha = \) angle counted on bearing’s central plane from the direction pointing to
- maximum film thickness in the steady state
- \( \alpha_i(\xi) \) = function which defines boundary of lubricant
- \( \beta = \) value of \( \nu \) on the vector \( \Omega \)
- \( \beta' = \) value of \( \beta \) in the steady state
- \( A \) = perturbation of \( \beta \) from \( \beta' \)
- \( \alpha = \) angle counted on bearing’s central plane from the direction pointing to
- \( \eta = \) constant viscosity of lubricant
- \( \theta, \psi, \varphi \) = Euler angles of \( \tau' \) with reference to \( \tau \)
- \( \Omega = \) center of bearing
- \( \omega = \) rotational speed of journal
- \( \tau = \) nondimensional time; \( \tau = \omega t \)

**Transactions of the ASME**
Applications

Formulas (10) and (11) are rather complex and will be fully exploited together in a future paper on the dynamics of rotors (Holmes has used formula (10) already in [1]). We will give here only two immediate applications of formula (11) to two asymptotic cases: The case of small $A$ of the first order and the case of $A$ near to 1. In the first case, formulas (11) reduce to

$$M_s = \frac{\eta R b \omega}{120 c^3} \left\{ \frac{\pi}{2} \left( 1 - 2 \psi' \right) \sin (\beta - \psi) + \pi \psi \cos (\beta - \psi) \right\},$$

$$M_s = \frac{\eta R b \omega}{120 c^3} \left\{ \pi \psi \sin (\beta - \psi) + \frac{\pi}{2} \left( 1 - 2 \psi' \right) \cos (\beta - \psi) \right\},$$

so that the components $M_1, M_2$ of $\mathbf{M}$ along the nodal line and in a direction normal to that line within the journal's central plane are simply

$$M_1 = -\frac{\pi \eta R b \omega}{120 c^3} \psi' \left( 1 - 2 \psi' \right),$$

$$M_2 = \frac{\pi \eta R b \omega}{240 c^3} \theta (1 - 2 \psi').$$

(12)

These formulas can be derived also from earlier developments [2, 3], when account is taken of the assumptions made here (incompressible lubricant, smooth bearing, full cavitation). They can be applied, for instance, to the study of stability of a rigid journal against conical whirl, as in [3]. Following the analysis of that paper with the specification (12) of $M_1, M_2,$ the rule of stability already found there can be confirmed; the condition of stability is purely geometrical: For stability, the moment of inertia $C_1$ of the journal around its axis must exceed the moment of inertia $C_1$ around an axis in the central plane, divided by two: $C_1 > C_1/2$.

In the second limit case of a journal under heavy load, in formulas (11) the terms prevail which contain the highest power of $1/(1 - A')^2$; it must be noticed also that, when $A$ tends to 1, $\beta$ tends to zero according to the formula

$$\tan \beta = \frac{\pi}{4} \sqrt{1 - A'^2},$$

because, under heavy load, the journal approaches the lowest permissible position.

Then, in approximation, $M_s$ and $M_s^*$ reduce to

$$M_s = -\frac{\eta R b \omega}{15 c^3} \left( 1 - A' \right)^{7/2} \theta \sin \psi,$$

$$M_s^* = -\frac{\eta R b \omega}{15 c^3} \left( 1 - A' \right)^{7/2} \theta \sin \psi,$$

and actually $M_s$ prevails over $M_s^*$. Therefore, the lubricant opposes most rigidly attempts to rock the journal around a horizontal axis ($\psi = \frac{\pi}{2}$) and the restoring couple acts then approximately as if caused by a spring of angular rigidity $\eta R b \omega c^{-7} (1 - A')^{-7/2}$. The journal is, instead, relatively free to swing around a vertical axis ($\psi = 0$); the moments excited in that case are approximately zero.

A similar behavior can be noticed in $F_s$; when $A$ tends to 1, formulas (10) yield the approximations

$$F_s = -\frac{4 \eta R b \omega}{c (1 - A')^{1/2}} a_0,$$

$$F_s^* = -\frac{3 \pi \eta R b \omega}{4 c (1 - A')^{5/2}} a_0.$$

Hence $F_s$ prevails over $F_s^*$ and the lubricant force is greatest in a radial direction, having then the characteristics of an elastic force of rigidity $4 \eta R b \omega c^{-7} (1 - A')^{-7}$. 

References