THE BUBBLY FLUID AS A CONTINUUM
WITH MICROSTRUCTURE

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1. Introduction

The approach to the dynamics of continua with microstructure suggested in [2] and [3] leads to general balance equations (for momentum, micromomentum and total moment of momentum) which comprise as special cases the equations proposed by various Authors to describe the behaviour of many particular types of generalized continua. For instance, if the state of the microstructure can be specified by a single scalar parameter \( \nu \), the equations in question are:

(I) the usual equation of mass balance for the continuous medium as a whole:

\[
\dot{\rho} + \rho \text{ div } \mathbf{x} = 0, \quad (1.1)
\]

(\( \rho \), density; \( \mathbf{x} \), velocity);

(II) the usual Cauchy equation, again for the continuum as a whole,

\[
\rho (\ddot{\mathbf{x}} - \mathbf{b}) = \text{div } \mathbf{T}, \quad (1.2)
\]

(\( \mathbf{b} \), body force per unit mass; \( \mathbf{T} \); Cauchy stress);

(III) a lagrangian equation of balance for the microstructure

\[
\rho (\mu \dot{\nu} + \frac{k_2}{2} \frac{d\nu}{d\tau} \dot{\nu}^2) = \rho \beta - \zeta + \text{div } \mathbf{t} \quad (1.3)
\]

(here \( \nu \) is a function of \( \nu \) such that \( \kappa = \frac{k_2}{2} \dot{\nu}^2 \) is the extra kinetic energy per unit mass due to the microstructure; \( \beta \), \( -\zeta \), \( \mathbf{t} \) describe actions on the microstructure:

\( \beta \), the external force per unit mass;
-\zeta, the internal force per unit volume;
 t-n, the surface force on an element of unit area and normal n); 

(IV) a global equation of balance for moment of momentum

\[ \text{skw}T = \frac{1}{2} \nabla \cdot (\zeta \mathbf{r} + \nabla \cdot r t); \]  

(1.4)

(here \( \nabla \) is the Ricci alternator and \( r \) is a vector such that

\[ \nabla(R) = r \cdot w \]  

(1.5)

is the time-rate of \( v \) in a global rigid motion of the body where the macroscopic velocity field is \( \mathbf{x} = \mathbf{a} + \mathbf{w} \times \mathbf{x} \).

Continua with voids and bubbly fluids can be studied with the help of (1.1) \( \div (1.4) \); the latter at least when the bubbles are carried with the fluid and only expansion and contraction of the bubbles is envisaged. Then \( v \) (\( v \in (0, 1) \)) can be interpreted as void (or bubble) fraction, a parameter which remains unchanged in a global rigid motion; thus \( r \) must be taken to be null in (1.5) with the conclusion that

\[ T \in \text{Sym}, \]  

(1.6)

under the circumstances (see also Section 10 of [3]).

Appropriate specification of \( \zeta \) and of the constitutive functions \( \mu, \zeta \) and \( t \) for bubbly fluids is now required; many Authors have addressed themselves to this task, using the theory of immiscible mixtures as a starting point, or, alternatively, relying on experiment and intuition (see, e.g., [1], [4], [8] and [9]). Particular reference must be made to [4], where the developments have a very general character but also lead to specific results. Here we examine the question again, in the spirit of the theory of continua with microstructure.
2. **Constitutive functions**

(I) **Kinetic energy.**

Within our context the continuous body $B$ is imagined to consist of volume elements, each containing one bubble and thus having as measure $\Psi$ the inverse of the number density of the bubbles. All that occurs within each element must be an attribute of the microstructure and the tools at our disposal to express the attributes are severely limited: $\nu$ and its derivatives.

To begin with, the geometric meaning of $\nu$ is made patent by the relation

$$\rho = \rho_0 (1 - \nu) + \rho_b \nu,$$

where $\rho$ and $\rho_0$ are the (local) densities of fluid and of bubble material (gas, vapour) respectively. If there are no exchanges (due to chemical reactions or vaporization), then the two species are separately conserved

$$\rho_0 (1 - \nu) \mathbf{i} = \rho_0^* (1 - \nu^*_b), \quad \rho_b \nu \mathbf{i} = \rho_b^* \nu_b^*. $$

here an asterisk indicates the value in the reference placement and

$$\mathbf{i} = \det \nabla \mathbf{x}. $$

An appropriate expression for the kinetic energy must now be decided upon; in general a fluid element can be scarcely modelled as a lagrangian system with a finite number of degrees of freedom but there are occasions when it can be treated that
way (see, e.g., [5], pp. 558-564). We think here of a spherical element of volume \( v \) with a concentric spherical bubble; within the element the gas is subject to homogeneous contractions and expansions, whereas the fluid moves radially without change in volume. The latter assumption implies that \( \rho_f \) is constant and, hence, (see (2.2) \( _1 \)) \( \nu \) and \( \iota \) are constrained by

\[
(1 - \nu) \iota = 1 - \nu \iota, \quad \text{and} \quad (1 - \nu) \nu = \nu \iota. \quad (2.4)
\]

Elementary calculations lead to the following expression for the kinetic energy per unit mass

\[
\kappa = \frac{\nu^2}{2} \left( \frac{\nu_0^2 (1 - \nu_0)}{48 \pi^2} \right)^{1/3} \frac{\rho_f^2 (1 - \nu^{1/3}) + \frac{1}{5} \rho_b}{\nu^{1/3} (1 - \nu)^{2/3} (\rho_f (1 - \nu) + \rho_b \nu)}. \quad (2.5)
\]

Of course, if, as is often the case, one can disregard \( \nu_0 \), \( \nu \) and \( \rho_b / \rho_f \) with respect to unity, then a well-known simpler expression suffices

\[
\kappa = \frac{1}{2} \alpha \nu^{-1/3} \nu, \quad \alpha = \left( \frac{\nu_0^2}{48 \pi^2} \right)^{1/3}. \quad (2.6)
\]

(II) Global stress

From a macroscopic point of view B is presumed to behave as a linearly viscous compressible fluid with shear viscosity \( \eta \) and bulk viscosity \( \chi \), so that

\[
T = (- w + \chi \text{ div } \dot{x}) 1 + 2 \eta (\text{sym grad } \dot{x} - \frac{1}{3} (\text{div } \dot{x}) 1), \quad (2.7)
\]

where, in general, the pressure \( w \) and the viscosities \( \chi \) and \( \eta \) may depend also on \( \nu \).

(III) Lagrangian forces

From (1.2), (1.3) the power of forces acting on B can be
calculated; in particular the power density of internal actions is found to be

\[ -(T \cdot \text{grad } \dot{x} + \zeta \dot{\nu} + t \cdot \text{grad } \dot{\nu}). \quad (2.8) \]

In fact it is the requirement that this density be null for any rigid velocity distribution that leads to condition (1.4).

Expression (2.8) offers also a clue as to appropriate constitutive assumptions for \( \zeta \) and \( t \). Observe that, on the one hand, in a uniform expansion (2.8) reduces to

\[ -\frac{1}{3} (\text{tr}T) l^{-1} l \cdot - \zeta \nu, \quad (2.9) \]

or, taking into account (2.7) and (2.4),

\[ \nu \frac{\dot{\nu}}{1 - \nu} - \chi (\frac{\dot{\nu}}{1 - \nu})^{2} - \zeta \nu. \quad (2.10) \]

On the other hand if the gas and fluid which make up each cell obey standard linear constitutive hypotheses, then

(I) there is no viscous loss and the pressure \( \bar{w}_g \) is uniform within the bubble; \( \bar{w}_g \) satisfies a relation of the type

\[ \bar{w}_g = \bar{w} \left( \frac{\rho_b}{\rho_f} \right) \gamma = \bar{w} \left( \frac{\nu_f}{1 - \nu_f} \right) \left( \frac{1 - \nu_f}{\nu_f} \right) \gamma, \quad (2.11) \]

where \( \bar{w} \) and \( \gamma \) are appropriate positive constants;

(II) the pressure does no work in the incompressible fluid but there are losses due to the shear viscosity \( \eta_f \);

(III) the surface tension introduces an energy proportional, by a constant factor \( \sigma \), to the surface of the bubble.

Thus the power density (2.10) is also found, through elementary calculations, to be equal to
\[ \frac{w}{g} \frac{\nu}{1 - \nu} - \frac{4}{3} \eta_f \frac{\nu^2}{\nu (1 - \nu)} - 2 \sigma \left( \frac{4\pi}{3\nu} \right) \frac{(1 - \nu)^2}{(1 - \nu^*) \nu^3} \nu \]  

(2.12)

Comparison of (2.12) with (2.10) prompts a remark (first advanced by G. I. Taylor): the bulk viscosity of the medium should be related to \( \eta_f \) as follows

\[ \chi = \frac{4}{3} \eta_f \frac{1 - \nu}{\nu}. \]  

(2.13)

Given (2.13), consistency is assured if

\[ \zeta = \frac{w - w_0}{1 - \nu} + 2 \sigma \left( \frac{4\pi}{3\nu} \right) \frac{(1 - \nu)^2}{(1 - \nu^*) \nu^3} \nu. \]  

(2.14)

The choices (2.13), (2.14) are not unique, but will be accepted from now on.

The question remains open as to an appropriate specification of \( t \) and \( \beta \). They seem to have both an important role only when the diffusion of bubbles in a fluid is discussed; thus, in view of our present limited goal, we will restrict our attention below to cases where they may be both regarded as null

\[ \beta = 0, \quad t = 0. \]  

(2.15)

Finally a decision must be made regarding the relation between \( \eta \) and \( \eta_f \): within the limits which allow us to accept (2.5), Einstein\'s formula can be presumed valid for \( \eta \)

\[ \eta = \eta_f (1 + \frac{5}{2} \nu); \]  

(2.16)

more generally the linear dependence on \( \nu \) could be substituted by a more complex one, adapted to the conditions which apply at higher concentrations (see, e.g., [5], Ch. 9).

3. The dynamics of dilute suspensions of bubbles.

Introduction of (2.7), (2.13) in (1.2) and of (2.6), (2.11),
(2.14), (2.15) in (1.3) leads to explicit balance equations
\[ \rho (\ddot{x} - b) = - \nabla \cdot (\mathbf{w} - \frac{4}{3} \eta_f \dot{\mathbf{v}}/\nu + \eta_f (1 + \frac{5}{2} \nu) (\Delta \ddot{x} + \frac{1}{3} \nabla \cdot \nabla \ddot{x}) + \nabla \cdot \nabla \text{div} \ddot{x}) + 5 \eta_f \text{sym} \nabla \ddot{x} - \frac{1}{3} (\text{div} \ddot{x}) \text{div} \mathbf{v}, \]  
(3.1)
\[ \rho a (v^2 \dot{\gamma} - \frac{1}{6} v v^2 - \frac{1}{6} v v^2) = (1 - \nu)^{-1} \left( \frac{\mathbf{w}}{1 - \nu^*} \left( \frac{1}{1 - \nu^*} \frac{1 - \nu}{\nu} \right) - \mathbf{w} \right) + \]
\[ - 2 \sigma \left( \frac{4 \pi}{3 \nu^*} \left( \frac{1 - \nu}{1 - \nu^*} \right)^2 \right) \mathbf{v}^3. \]  
(3.2)

The equation of balance of mass (1.1) and the constraint (2.4) complete the set
\[ \rho' + \rho \text{div} \ddot{x} = 0, \]  
(3.3)
\[ \text{div} \ddot{x} = \frac{\dot{\mathbf{v}}}{1 - \nu}. \]  
(3.4)

The considerations which have led us to this set are, in part, purely suggestive; however, consistency is assured. In fact: (I) condition (1.6) is obviously satisfied; (II) \( \mathbf{w} \) is the correct (i.e., workless) reaction to the constraint (3.4); (III) the choices of \( T, \mu, \zeta, \tau \) and \( \beta \), though special, do not conflict with any general principle. If the reaction \( \mathbf{w} \) is eliminated from (3.1) through (3.2) and \( \nu, \dot{\nu} \) are expressed in terms of \( Vx, \text{div} \dot{x} \), eqn (3.1) takes the form of a balance equation where the stress involves, beyond the usual terms typical of an elastic viscous fluid, also virtual inertia terms, as in formula (7.17) of [7].

A glimpse at the complexity of the behaviour of solutions of (3.1), (3.2), (3.3), (3.4) can be had already by examining possible static solutions under gravity (\( b = g, \text{const} \)), when
\[ \mathbf{w} = \mathbf{w}_{\text{atm}} + \rho |g| \xi \]
(\xi, \text{ depth from free boundary, where } \bar{w} = w_{\text{atm}}) \text{ and } v \text{ must satisfy}

\bar{w} = \bar{w} \left( \frac{\nu_*}{1 - \nu} \frac{1 - v}{\nu} \right)^{\frac{1}{2}} - 2\sigma (1 - \nu) \frac{4\pi}{3\nu_*} \left( \frac{1 - v}{1 - \nu_*} \right)^{\frac{1}{2}}. \quad (3.5)

Eqn (3.5) need not admit in general a unique solution in \nu, though the values usually taken by the parameters may lead to uniqueness.

Observe also that solenoidal solutions, if they exist, are those possible in a special viscous fluid, with \nu and \bar{w} constant on each particle and related by (3.5).

For one-dimensional flows, (3.3), (3.1), (3.2) reduce to (4.7), (4.8), (4.9) respectively of [9], provided that \eta and \sigma are put equal to zero; the only inessential discrepancy is in the position of the 'bulk viscosity' term, an ambiguity on which we have already remarked.

To complete the comparison with the developments of [9] we report below the linearized equations which obtain when \beta = 0 and \nu = \dot{x} is small of the first order, together with the variations \partial / \partial \nu, \varepsilon and \partial \bar{w} / \partial \nu of \rho, \nu and \bar{w} respectively from static constant values \rho_*, \nu_* and \bar{w}_*, the latter two related by

\bar{w}_* = \bar{w} - 2\sigma \left( \frac{4\pi}{3\nu_*} \frac{1 - \nu_*}{\nu_*} \right)^{\frac{1}{2}}. \quad (3.6)

The equations are:

\[ \rho_* \frac{\partial \nu}{\partial \tau} = - \text{grad} \left( \bar{w}_* \delta - \frac{4}{3} \eta \left( \frac{1 - \nu_*}{\nu_*} \right) \text{div} \nu \right) + \eta \left( \Delta \nu + \frac{1}{3} \text{grad} \text{div} \nu \right), \]

\[ \frac{\partial^2 \bar{w}}{\partial \tau^2} = \frac{(\bar{w} - \bar{w}_*) (4 + \nu_* \bar{w}) - 3\gamma \bar{w}}{3\nu_* (1 - \nu_*)^2 \rho_* \alpha} \varepsilon - \left( \frac{\bar{w} \nu_*}{(1 - \nu_*) \rho_* \alpha} \right)^{\frac{1}{2}}. \quad (3.7) \]
\[ \frac{\partial v}{\partial t} + \rho \text{div} v = 0, \]

\[ \frac{\partial \varepsilon}{\partial t} = (1 - \nu) \text{div} v. \]

Eqs (3.7) can be compacted in a single equation in \( v \)

\[
\frac{\partial^2 v}{\partial t^2} = \text{grad div} \left( \frac{(1 - \nu)^2 \alpha}{\nu^{1/2}} \frac{\partial^2 v}{\partial t^2} + \frac{\eta_f}{3 \rho \nu} (4 - 3 \nu) \frac{\partial v}{\partial t} + \right. \\
+ \left. \gamma\tilde{w} - \frac{1}{3} (1 + 4 \nu) \left( \tilde{w} - \tilde{w}_* \right) \frac{\partial^2 v}{\partial t^2} \right) + \frac{\eta_f}{\rho \nu} \frac{\partial^2 v}{\partial \nu} 
\] (3.8)

(see (4.11) of [9], where \( \eta_f = 0, \gamma = 1, \tilde{w} = \tilde{w}_* \)). If \( \eta_f \) vanishes, sinusoidal waves of constant amplitude, length \( \lambda \) and speed \( \psi \) are admitted by (3.9), provided \( \lambda \) and \( \psi \) satisfy the dispersion relation

\[
\psi^2 = \frac{[\gamma\tilde{w} + \frac{1}{3} (1 + 4 \nu) \left( \tilde{w}_* - \tilde{w} \right)] \lambda^2}{\rho \nu \alpha^2 + 4 \pi^2 (1 - \nu) \nu^{1/2} \alpha}.
\] (3.10)

Thus the frequency \( 2\pi \psi/\lambda \) must be less than

\[
\frac{\gamma\tilde{w} + \frac{1}{3} (1 - 4 \nu) \left( \tilde{w}_* - \tilde{w} \right)}{\rho \nu (1 - \nu) \nu^{1/2} \alpha},
\]

this critical value appears already in other studies of continua with voids.

Acceleration waves may propagate in the fluid only in a degenerate case when \( \eta_f = 0 \) and \( \alpha = 0 \), i.e. in a case when the virtual inertia due to the expansion and contraction of the bubbles can be disregarded, whereas the bounce provided by the gas in the bubbles is still relevant.
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REFERENCES


