A Quasi-Stationary Phase-Field Model with Micro-Movements

Giulio Schimperna  

*Dipartimento di Matematica, Università di Pavia  
via Ferrata 1, I-27100 Pavia, Italy  
e-mail: giulio@dimat.unipv.it*

Ulisse Stefanelli  

*Istituto di Matematica Applicata e Tecnologie Informatiche – CNR  
via Ferrata 1, I-27100 Pavia, Italy  
e-mail: ulisse@imati.cnr.it*

Abstract

This note addresses the global strong solvability of a phase-field system arising in connection with the phase transition theory recently proposed by M. Frémond [10]. The novelty of this modelization consists in considering the macroscopic effect of the microscopic movements of particles of the system that undergoes the phase transition. In particular, we shall outline the basic features of this model and deal with the upcoming nonlinear PDE system in the one-dimensional setting by means of an approximation – a priori estimates – passage to the limit procedure.

**Key words:** phase transition, microscopic movements, phase-field, irreversibility.

**AMS (MOS) Subject Classification:** 80A22, 35K55.

1 Introduction

In the framework of Continuum Thermo-mechanics, M. Frémond has recently proposed a new approach to the modeling of phase transition processes [10]. The main novelty of such an approach relies in allowing for *macroscopic movements* of particles by carefully taking into account their effect on the overall macroscopic behavior of the body. In particular, although the substance that undergoes the phase transition may often be macroscopically regarded as a *rigid continuum*, it is clear that, in order the phase transition to occur, no rigidity can be assumed at the microscopic level. In fact, the phase transition is, heuristically speaking, the effect of some structural rearrangement or reorganization of the microscopic components of the substance. We shall briefly outline some points of this modeling perspective in the forthcoming Section 2. Let us however stress here that the key feature of the above mentioned models for phase transitions with microscopic movements (micro-movements) is that the *thermodynamic consistency* can be proved. This is indeed a crucial point since it is possible to include in the latter generalized framework a variety of classic models of phase transition [22]. In particular, we
shall at least mention the Stefan model, the phase relaxation model [11, 23], and some Penrose-Fife phase-field models [20, 21]. Moreover, although beyond the purposes of this analysis, some connections may also be given with the classic Caginalp phase-field model [8]. In this regards, Frémond’s modelistic framework entails somehow a unifying approach to the derivation and the thermodynamic justification of the above mentioned classic models.

The present contribution addresses the study of a quasi-stationary phase-field model that arises in the framework of phase transitions with micro-movements. In particular, referring indeed to the forthcoming Section 2 for details and motivation, we shall be concerned with the evolution of two unknowns \( \vartheta, \lambda \) governed by the relations

\[
\begin{align*}
\vartheta_t + \vartheta \lambda_x &= \vartheta_{xx}, \\
-\nu \lambda_{xx} + \beta(\lambda) &\geq \vartheta - \vartheta_c,
\end{align*}
\]  

(1.1) (1.2)

in the space-time domain \((0, 1) \times (0, T)\), for some reference time \(T > 0\). Let us mention that the unknowns \( \vartheta, \lambda \) shall denote the absolute temperature of the medium and an order parameter, respectively. Moreover, \( \beta \subset \mathbb{R} \times \mathbb{R} \) is a maximal monotone graph, possibly multivalued, the parameter \( \nu > 0 \) is fixed, and \( \vartheta_c > 0 \) is a constant standing for the transition temperature. We shall remark that an interesting choice for \( \beta \) turns out to be given by the constraint \( \beta = \partial I_{[0,1]}\). The latter is nothing but the well-known sub-differential of the indicator function \( I_{[0,1]} \) of the interval \([0,1]\), i.e. \( I_{[0,1]}(r) = 0 \) in \( r \in [0,1] \) and \( I_{[0,1]}(r) = +\infty \) otherwise. In particular, referring to [6] for details, we stress that \( y \in \partial I_{[0,1]}(x) \) iff \( x \in [0,1] \) and \( y(x - z) \geq 0 \) for all \( z \in [0,1] \). Hence the choice \( \beta = \partial I_{[0,1]} \) forces the order parameter \( \lambda \) to attain values in \([0,1]\) and is especially motivated in case \( \lambda \) represents a local proportion of one phase against the other. Finally, we shall complement (1.1)-(1.2) with suitable initial conditions and with homogeneous Neumann boundary conditions. We stress that, although the choice of the latter boundary condition is strongly motivated from the physics of the problem, we would actually be in the position of considering different and possibly more general situations.

Despite the one-dimensional setting, the study of (1.1)-(1.2) is in our opinion quite challenging. First of all, the presence of a nonlinearity in (1.1) represents a novelty with respect to former contributions on phase-field models. Secondly, the system (1.1)-(1.2) turns out to be the first relaxation approach to the Stefan model in the framework of phase transitions with micro-movements. Indeed, by setting \( \nu = 0 \) in (1.2) and restricting to the above introduced case \( \beta = \partial I_{[0,1]} \), we readily check that (1.1)-(1.2) turns out to be equivalent to a singular parabolic equation of the form

\[
(\vartheta + \lambda(\vartheta))_t = \vartheta_{xx}.
\]  

(1.3)

In the latter, \( \lambda \) is a multivalued graph that presents a vertical segment in correspondence of the transition temperature \( \vartheta_c \) and will be detailed in the forthcoming Section 2. It is well-known that equation (1.3) is one of the weak formulations of the two-phase Stefan model. Let us however stress from the very beginning that we will not provide a detailed justification of the latter formal asymptotics since this seems an indeed very challenging task.

The main contribution of the present analysis is that of providing the global existence of a strong solution to (1.1)-(1.2). We exploit, with this aim, an approximation procedure. In particular, both the nonlinear graph \( \beta \) and the initial data are replaced by suitable approximations and an existence result for the regularized problem is established. Then, we use both compactness and monotonicity techniques in order to pass to the limit in the approximation and prove the existence of a strong solution for the original problem. Moreover, we are in the position of providing a global solvability result for some more general systems where we retain
(1.1) and replace (1.2) by
\[ \alpha(\chi) - \nu \chi_{xx} + \beta(\chi) \ni \vartheta - \vartheta_c. \]  
(1.4)
Here \( \alpha \subset \mathbb{R} \times \mathbb{R} \) is another maximal monotone graph, possibly multivalued. Namely the problem turns out to be \textit{doubly nonlinear} and multivalued in the order parameter dynamics. Our analysis will apply to a large class of choices for the graph \( \alpha \) among which we shall mention at least \( \alpha = \text{identity} \) and \( \alpha = \partial I_{[0, +\infty)} \). The first one actually gives rise to a full phase-field model. However let us remark that the system (1.1), (1.4) cannot be completely justified in the framework of our physical derivation (see (1.5)-(1.6) below and Section 2). As for the second, we have to recall that \( y \in \partial I_{[0, +\infty)}(x) \) iff \( x \in [0, +\infty) \) and \( y(x-z) \geq 0 \) for all \( z \in [0, +\infty) \). In particular, the choice \( \alpha = \partial I_{[0, +\infty)} \) forces \( \chi_t \) to attain nonnegative values. We will refer to the latter situation as that of an \textit{irreversible} phase transition, meaning in particular that \( \chi \) increases along the trajectories of the system and with no explicit relation with the concept of irreversibility in Thermodynamics.

We shall now recall some related contributions on the modeling of phase transitions with micro-movements. In particular, let us mention the two papers [13] and [18] where the \textit{full phase-field models}
\begin{align*}
\vartheta_t + \vartheta \chi_t &= \vartheta_{xx} + \mu \chi_t^2, \\
\mu \chi_t + \alpha(\chi_t) - \nu \chi_{xx} + \beta(\chi) &\ni \vartheta - \vartheta_c,
\end{align*}
(1.5)
(1.6)
with \( \mu > 0 \), are investigated in the case \( \alpha = \partial I_{[0, +\infty)} \) and \( \alpha = 0 \), respectively. In the above mentioned papers, the main result of the analysis is the proof of the existence of a strong global solution for some related initial and boundary value problem. The analytical techniques that are devised in [13] and [18] cannot be easily adapted to the current \textit{quasi-stationary} situation of (1.1)-(1.2) where indeed \( \mu = 0 \). In this connection some \textit{ad hoc} techniques need to be developed. The reader shall also be referred to the papers [9, 15, 16, 17] where the analysis of some three-dimensional phase transition problem within the framework of micro-movements is presented. See also [4, 5] and the thesis [22] for other related results.

This is the plan of the paper: we shall briefly discuss the derivation of the above PDE systems in Section 2. Then, notations and the complete statement of our main result are contained in Section 3, while its proof is provided in Section 4. In particular, we will address a regularized problem in Subsection 4.1, establish some a priori estimates in Subsection 4.2, and finally pass to the limit in Subsection 4.3.

## 2 Derivation of the model

We shall collect here, for the reader’s convenience, some introductory discussion on Frémond’s novel approach to the macroscopic modeling of phase transitions. The tracement will be developed just in the direction of motivating the previously introduced PDE systems. In particular, we are not interested in giving the justification of the model in its full generality. For this issue and all the additional details the reader is referred at once to the monograph [10].

Assume we are given a substance which may undergo a two-phase transformation and may be assimilated to the one-dimensional region \( (0, 1) =: \Omega \). We aim to study the evolution of the system in a fixed time interval \( [0, T] \) where \( T > 0 \) is a final reference time.

In order to describe the system we will focus on the two state variables \( \vartheta \) and \( \chi \). Namely, the first variable \( \vartheta \) stands for the \textit{absolute temperature} of the body, hence it is assumed for
the purposes of this section to be strictly positive (this positivity will turn out to be a theorem later on). On the other hand, \( \chi \) represents an order parameter describing the phase transition. The latter is possibly to be interpreted as a local proportion between the different phases. In particular, following this interpretation, \( \chi \) is supposed to range in the interval \([0, 1]\) where the extreme values are regarded as pure phases while the situation \( \chi \in (0, 1) \) stands for a possible mixture of them (a so-called mushy region).

The main novelty of the modeling relies in assuming that, although in a rigid-body framework, the microscopic movements of molecules give rise to macroscopic effects. To this end, let us suppose from the very beginning that the proper quantities describing the microscopic movements of particles are \( \dot{\chi} \) and \( \dot{\chi}_x \), where of course the dot indicates differentiation with respect to time.

In this context, it turns out to be convenient to regard the vector \( (u, \dot{\chi}) \) (where \( u = u(x, t) \) is the velocity of the material point \( x(t) \)) as an actual rigid velocity vector. Moreover, let us assume that there exists a linear space of virtual rigid body velocities \( R \) (see [10] for a full discussion). Finally, we assume that, for all times \( t \in [0, T] \), the virtual power of the internal forces of the body with respect to the subdomain \( D = (d_1, d_2) \subset \Omega \) and the virtual velocities \( (v, c) \in R \) is

\[
P_{\text{internal}}(D, v, c) := \int_D (Bc + Hc_x) \, dx.
\]

In the latter relation the two quantities \( B \) and \( H \) come into play. Indeed, an obvious dimensional argument entails that they shall be regarded as energy densities. In particular, \( B \) represents an energy density per unit of \( \chi \), while \( H \) may be considered as a density of energy flux. Moreover, we observe that the term \( \int_D \sigma v_x \), where \( \sigma \) is the stress, does not appear above. This is an effect of our overall rigid body assumption.

We shall now introduce the virtual power of the external forces as

\[
P_{\text{external}}(D, v, c) := \int_D (fv + Ac) \, dx + (Tv + ac)(d_2) - (Tv + ac)(d_1).
\]

In the latter, the terms involving \( v \) are quite classical and \( f \) represents an action density at distance (body force) while \( T \) is a action density at contact (traction). On the other hand, the quantity \( A \) is a volume density of energy supplied to the material per units of \( \chi \) by microscopic actions without macroscopic motions. One may think, for instance, to chemical or radioactive actions. Moreover, the quantity \( a \) represents the surface density of energy per units of \( \chi \) supplied to the body.

Finally, as for the virtual power of acceleration forces we set

\[
P_{\text{acceleration}}(D, v, c) := \int_D \rho \gamma v \, dx,
\]

where \( \gamma = \dot{u} \) is the acceleration and \( \rho \) is the (constant) material density. In the latter position the acceleration related to microscopic movements has been neglected. For a contribution in the direction of including such an acceleration the reader is referred to [5].

Let us now recall the Virtual Power Principle [12]. Namely, for all subdomain \( D = (d_1, d_2) \subset \Omega \), and all virtual rigid body velocities \( (v, c) \in R \), we impose that

\[
P_{\text{acceleration}}(D, v, c) = P_{\text{internal}}(D, v, c) + P_{\text{external}}(D, v, c).
\]
We assume from the very beginning that $A = a = 0$ since the mathematical treatment of non-zero data is straightforward. Hence, the latter relations turn out to imply, in particular, the equation of motion

$$B = H_x \text{ in } \Omega \times (0, T), \quad H = 0 \text{ in } \{0, 1\} \times (0, T).$$

(2.1)

We now introduce the energy balance relation. Letting $e$ denote the internal energy density of the body, $q$ the heat flux and supposing that the system is insulated from the exterior we can follow [10, Sec. 3.2] and deduce that, in our situation, the First Principle of Thermodynamics is expressed by the relations

$$\dot{e} + q_x = B \dot{\chi} + H \dot{x}_x \text{ in } \Omega \times (0, T),$$

$$q = 0 \text{ in } \{0, 1\} \times (0, T).$$

(2.2) \hspace{1cm} (2.3)

In particular, we note that the right hand side of (2.2) differs from zero and takes into account the contribution to the energy balance of the micro-movements.

The next step is to define the quantities $e, q, B, H$ in terms of the state variables in such a way that the Second Principle of Thermodynamics, in the form of the Clausius-Duhem inequality, is pointwise fulfilled. In particular, the latter reduces in our case to

$$\dot{S} + \frac{q}{\theta} \leq 0,$$

(2.4)

where $S$ is the entropy of the system. In order to accomplish this requirement we will exploit the Ginzburg-Landau theory by introducing the free energy density function $\Psi = \Psi(\theta, \chi, x_x)$ and defining

$$S := -\frac{\partial \Psi}{\partial \theta}, \quad e := \Psi + \theta S.$$

As for the heat flux we choose the standard Fourier law

$$q := -k \partial x_x,$$

where $k$ is a positive conductivity constant.

Moreover, we will introduce a pseudo-potential of dissipation $[10, 19]$ $\Phi = \Phi(\dot{\chi})$, which is a convex and non-negative function vanishing in zero, and define

$$B := \frac{\partial \Psi}{\partial \chi} + \frac{\partial \Phi}{\partial \chi}, \quad H := \frac{\partial \Psi}{\partial x_x}.$$

Hence, let us go back to (2.2) and exploit the latter choices in order to get that

$$0 = (\Psi + \theta S) + q_x - B \dot{\chi} - H \dot{x}_x = \dot{\theta} S + q_x - \frac{\partial \Phi}{\partial \chi} \leq \dot{\theta} S + q_x,$$

where we used the properties of the pseudo-potential $\Phi$. Namely, as $\dot{\theta} > 0$ and the Fourier law holds, the Clausius-Duhem inequality (2.4) easily follows. Taking into account the above computation we conclude that, and as long as $\Phi$ is convex, non-negative, and vanishes in zero, the coupling of (2.1) and (2.2)-(2.3) gives rise to a thermodynamically consistent model.

We now come to our actual choice of $\Psi$. In particular, let

$$\Psi(\theta, \chi, x_x) = -c_s \theta \ln \theta - \frac{L}{\theta_c} (\theta - \theta_c) \chi + \frac{\nu}{2} |x_x|^2 + \hat{\beta}(\chi).$$

(2.5)
In the latter expression the first term is purely caloric and \(c_a\) represents a specific heat density, while the second term stands for the phase-temperature interaction, with \(L\) denoting the latent heat density and \(\vartheta_c\) the phase transition temperature. On the other hand, \(\frac{1}{2}|x|^2\) measures the interface energy and \(\nu\) is a positive constant standing for an interface energy density. As for the last term, \(\hat{\beta} : \mathbb{R} \to [0, \infty)\) represents a convex, proper, and lower semicontinuous potential on \(\chi\). For instance, we may consider the constraining term \(\hat{\beta} = I_{[0,1]}\) which forces the order parameter \(\chi\) to attain solely values in \([0,1]\).

As for the pseudo-potential of dissipation we choose

\[
\Phi(\dot{\chi}) := \ell I_{[0, +\infty)}(\dot{\chi}),
\]

where \(\ell = 1\) or 0 activates, or not, the constraining term \(I_{[0, +\infty)}(\dot{\chi})\). The choice \(\ell = 1\) actually forces \(\dot{\chi}\) to be non-negative, modeling indeed the situation of irreversible phase transitions.

Moving from the above positions, one exploits (2.1) and (2.2) in order to get

\[
e_a \dot{\vartheta} + L \dot{\chi} - k \vartheta_{xx} = -\frac{L}{\vartheta_c}(\vartheta - \vartheta_c)\dot{\chi} + \eta \dot{\chi}, \quad (2.7)
\]

\[
\eta - \nu \chi_{xx} + \xi = \frac{L}{\vartheta_c}(\vartheta - \vartheta_c), \quad (2.8)
\]

\[
\eta \in \ell \alpha(\dot{\chi}), \quad \xi \in \beta(\chi). \quad (2.9)
\]

Let us stress that, whenever \(\ell \alpha = \partial I_{[0, +\infty)}\) (or \(\ell \alpha \equiv 0\), of course) the term \(\eta \dot{\chi}\) in the right hand side of (2.7) is vanishing since \(\eta \neq 0\) only if \(\dot{\chi} = 0\). Namely, the model (2.7)-(2.9) reduces to (1.1), (1.4) up to the normalization of some constants.

Finally, we are now in the position of making precise relation (1.3). By setting \(\nu = \ell = 0\) and \(\hat{\beta} = I_{[0,1]}\) in (2.7)-(2.9) we readily get the system

\[
e_a \dot{\vartheta} + L \dot{\chi} = -\frac{L}{\vartheta_c}(\vartheta - \vartheta_c)\dot{\chi}, \quad \partial I_{[0,1]}(\chi) \ni \frac{L}{\vartheta_c}(\vartheta - \vartheta_c).
\]

It is now clear that \(\dot{\chi} \neq 0\) only if \(\vartheta = \vartheta_c\). Namely, owing to the inclusion, the right hand side of the first relation above vanishes and the latter system may be equivalently rewritten as

\[
e_a \dot{\vartheta} + L \dot{\chi} = k \vartheta_{xx}, \quad \partial I_{[0,1]}(\chi) \ni \frac{L}{\vartheta_c}(\vartheta - \vartheta_c),
\]

which is nothing but the standard variational formulation of the well-known two-phase Stefan problem.

## 3 Statement of the results

Let us start by recalling and setting some notations. Let \(\Omega := (0,1), \ T > 0, \ Q_t := \Omega \times (0,t), \) for any \(t \in (0,T]\), and \(Q := Q_T\). Moreover, we set

\[
H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{u \in H^2(\Omega) : u_\alpha(0) = u_\alpha(1) = 0\},
\]

endowed with the usual scalar products. The reader is referred to [14] for definitions and properties of Sobolev spaces. Let also \((\cdot, \cdot), \ | \cdot |\) denote the scalar product and the norm in
H, let \( \| \cdot \| \) be the norm in \( V \), and \( \| \cdot \|_E \) denote the norm in the generic normed space \( E \). Finally, given a convex function \( \varphi : \mathbb{R} \to [0, +\infty] \), we make use of the symbol \( \varphi^* \) to indicate its conjugate, namely
\[
\varphi^* : \mathbb{R} \to (-\infty, +\infty] \quad \varphi^*(x) := \sup_{y \in \mathbb{R}} (xy - \varphi(y)).
\]

We are now in the position of stating our assumptions:

(A1) \( \vartheta_c > 0 \) is a given constant.

(A2) \( \hat{\alpha} : \mathbb{R} \to [0, +\infty] \) is a proper, convex, and lower semicontinuous function such that
\[\min \hat{\alpha} = \hat{\alpha}(0) = 0, \quad \alpha := \partial \hat{\alpha}.\]

(A3) \( \hat{\beta} : \mathbb{R} \to [0, +\infty] \) is a proper, convex, and lower semicontinuous function such that
\[\min \hat{\beta} = \hat{\beta}(0) = 0, \quad \beta := \partial \hat{\beta}, \quad \text{and there exist two positive constants } C_1, C_2 \text{ such that}
\]
\[
\hat{\beta}(r) \geq C_1 r^2 - C_2 \quad \forall r \in D(\hat{\beta}),
\]

where \( D(\hat{\beta}) := \{ r \in \mathbb{R} : \hat{\beta}(r) < +\infty \} \) stands for the effective domain of \( \hat{\beta} \).

(A4) \( \vartheta_0 \in V, \vartheta_0 > 0 \) in \( \Omega \).

(A5) \( \chi_0 \in W, \chi_0 \in D(\beta) \) a.e. in \( \Omega \), \( \hat{\beta}(\chi_0) \in L^1(\Omega) \), and there exists \( \xi_0 \in H \) such that
\[
\xi_0 \in \beta(\chi_0) \text{ a.e. in } \Omega, \quad \text{and } \hat{\alpha}^*(\vartheta_0 - \vartheta_c + \chi_0, \chi_0 - \xi_0) \in L^1(\Omega).
\]

**Remark 3.1.** We notice that, in the particular case \( \alpha = 0 \ (\alpha = \partial I_{[0, +\infty)} \), the last integrability assumption in (A5) means that \( \vartheta_0 - \vartheta_c + \chi_0, \chi_0 - \xi_0 = 0 \) (\( \leq 0 \), respectively) almost everywhere in \( \Omega \).

Our existence result reads as follows

**Theorem 3.2.** Assume (A1)-(A5). Then, there exists a quadruple of functions \((\vartheta, \chi, \eta, \xi)\) such that
\[
\vartheta \in H^1(0, T; H) \cap C^0([0, T], V) \cap L^2(0, T; W),
\]
\[
\chi \in H^1(0, T; V) \cap L^\infty(0, T; W),
\]
\[
\eta \in L^\infty(0, T; H),
\]
\[
\xi \in L^\infty(0, T; H),
\]

and the following relations
\[
\vartheta_t + \vartheta \chi_t = \vartheta_{xx} \quad \text{in } Q,
\]
\[
\eta - \chi_{xx} + \xi = \vartheta - \vartheta_c \quad \text{in } Q,
\]
\[
\eta \in \alpha(\chi_t) \quad \text{in } Q,
\]
\[
\xi \in \beta(\chi) \quad \text{in } Q,
\]
\[
\vartheta(\cdot, 0) = \vartheta_0 \quad \text{in } \Omega,
\]
\[
\chi(\cdot, 0) = \chi_0 \quad \text{in } \Omega,
\]

hold almost everywhere. Moreover, letting \( b(t) := \| \chi_t \|_{L^1(0, t; L^\infty(\Omega))} \) for \( t \in [0, T] \), we have that
\[
\min \vartheta_0 e^{-b(t)} \leq \vartheta(x, t) \leq \max \vartheta_0 e^{b(t)}, \quad \forall (x, t) \in Q.
\]
The proof of this result will be carried out in the next section by exploiting a regularization procedure. Then, suitable a priori estimates are established and the passage to the limit is obtained via compactness and monotonicity arguments. Moreover, it will be clear that the regularized problem admits a unique solution. On the other hand, we are in the position of proving indeed the following (very reductive) uniqueness result.

**Proposition 3.3.** Let $\vartheta_0, \chi_0, \xi_0 : \Omega \to \mathbb{R}$ be given. Moreover, let $\beta$ be Lipschitz continuous. Then there exists at most one quadruple $(\vartheta, \chi, \xi, \eta)$ fulfilling (3.2)-(3.11).

We omit the proof of the latter result since it is essentially contained in the forthcoming fixed point argument for the regularized problem.

4 Proof of Theorem 3.2

We plan to address a regularization of the problem (3.6)-(3.11). Indeed, let us start by applying a regularization procedure to the graph $\beta$. Namely, for $\varepsilon \in (0,1)$, let $\beta_\varepsilon$ be the Yosida approximation of $\beta$ (we refer to [6] for details) and, consequently, denote by $\hat{\beta}_\varepsilon$ the unique primitive of $\beta_\varepsilon$ verifying $\hat{\beta}_\varepsilon(0) = 0$. It is well-known that

$$\hat{\beta}_\varepsilon(r) = \min_{s \in D(\beta)} \left( \frac{1}{2\varepsilon} |r - s|^2 + \hat{\beta}(s) \right).$$

Thus, we readily have that

$$\hat{\beta}_\varepsilon(r) \leq \hat{\beta}(r) \quad \forall r \in \mathbb{R}.$$  \hspace{1cm} (4.2)

Moreover, taking into account the coercivity assumption in (3.1), the function $\hat{\beta}_\varepsilon$ turns out to be coercive as well for sufficiently small $\varepsilon$. Namely, we have that

$$\hat{\beta}_\varepsilon(r) \geq \frac{C_1}{2} r^2 - C_2 \quad \forall r \in \mathbb{R}, \forall \varepsilon \in (0,1/(2C_1)).$$  \hspace{1cm} (4.3)

Indeed, let us consider $r \in \mathbb{R}$, $s \in D(\beta)$, and $\varepsilon \in (0,1/(2C_1))$. Then,

$$\frac{C_1}{2} r^2 \leq C_1 |r - s|^2 + C_1 s^2$$

$$\leq \frac{1}{2\varepsilon} |r - s|^2 + C_1 s^2 - C_2 + C_2 \leq \frac{1}{2\varepsilon} |r - s|^2 + \hat{\beta}(s) + C_2,$$

and the assertion (4.3) follows.

Moreover, we replace $\alpha$ in (3.8) with $\alpha_\varepsilon := \varepsilon \cdot \text{id} + \alpha$ (id denotes the identity in $\mathbb{R}$). Let us remark that the inverse graph $\alpha_\varepsilon^{-1}$ is Lipschitz continuous with Lipschitz constant $\varepsilon^{-1}$, at most. We also need to regularize properly the initial data. To this aim, we set $\eta_0 := \vartheta_0 - \vartheta_c + \chi_{0,xx} - \xi_0$, we note that $\eta_0 \in H$ by (A4)-(A5) and, for all $\varepsilon \in (0,1)$, we introduce $\eta_{0,\varepsilon}$ as the solution of the following singular perturbation problem

$$\eta_{0,\varepsilon} \in W, \quad \eta_{0,\varepsilon} - \varepsilon \eta_{0,\varepsilon,xx} = \eta_0 \quad \text{in } \Omega.$$  \hspace{1cm} (4.4)

Next, let us approximate also $\chi_0$ by defining $\chi_{0,\varepsilon}$ as the solution of the problem

$$\chi_{0,\varepsilon} \in W, \quad -\chi_{0,\varepsilon,xx} + \beta_\varepsilon(\chi_{0,\varepsilon}) = -\eta_{0,\varepsilon} + \vartheta_0 - \vartheta_c \quad \text{in } \Omega.$$  \hspace{1cm} (4.5)
which exists and is unique by monotonicity of $\beta_\varepsilon$ and (4.3) (cf. [2]). Finally, we set $\xi_{0,\varepsilon} := \beta_\varepsilon(x_0,\varepsilon)$ and state a result whose formulation and proof follow with some modifications [3, Lemma 3.1].

**Lemma 4.1.** The approximated data $\eta_{0,\varepsilon}$, $x_{0,\varepsilon}$, and $\xi_{0,\varepsilon}$ satisfy the following properties:

1. For any $\varepsilon \in (0,1)$, we have that
   
   $\eta_{0,\varepsilon} \in W$, $x_{0,\varepsilon} \in H^3(\Omega)$, $\xi_{0,\varepsilon} \in V$. (4.6)

2. The family $-x_{0,\varepsilon,xx} + \xi_{0,\varepsilon}$ is bounded in the norm of $H$, uniformly in $\varepsilon$. Moreover, as $\varepsilon \to 0$ the following convergences hold

   $\xi_{0,\varepsilon} \to \xi_0$ weakly in $H$, $x_{0,\varepsilon} \to x_0$ weakly in $W$, $\eta_{0,\varepsilon} \to \eta_0$ strongly in $H$. (4.7)

3. There exists a real valued function $r(\varepsilon)$ such that $r(\varepsilon) \to 0$ as $\varepsilon \to 0$ and

   $$\int_{\Omega} \beta_\varepsilon(x_{0,\varepsilon}) \leq \int_{\Omega} \beta(x_0) + r(\varepsilon) \quad \forall \varepsilon \in (0,1).$$

4. There exists $C_3 > 0$ independent of $\varepsilon$ and such that

   $$\int_{\Omega} \alpha^*_\varepsilon(\eta_{0,\varepsilon}) \leq C_3 \quad \forall \varepsilon \in (0,1).$$

**Proof.** Property (P1) is a straightforward consequence of well-known regularity results for elliptic equations with Lipschitz nonlinearities. Similarly, (P2) may be proved by standard a priori estimates - compactness techniques. Note that the strong convergence in (P2) is a consequence of a semicontinuity argument applied to (4.4) tested by $\eta_{0,\varepsilon}$ and of, e.g., [2, Prop. 1.4, p. 12].

Now, let us show (P3). By definition of sub-differential (4.2), for all $\varepsilon \in (0,1)$ we have

$$\int_{\Omega} \beta_\varepsilon(x_{0,\varepsilon}) \leq \int_{\Omega} \beta_\varepsilon(x_0) + \int_{\Omega} (x_{0,\varepsilon} - x_0) \xi_{0,\varepsilon}$$

$$\leq \int_{\Omega} \beta(x_0) + |x_{0,\varepsilon} - x_0| |\xi_{0,\varepsilon}|,$$

where of course the latter quantity, which we can take as $r(\varepsilon)$, tends to 0 as $\varepsilon \to 0$ thanks to (4.7) (we use in particular the strong convergence $x_{0,\varepsilon} \to x_0$ holding, e.g., in $V$).

Finally, let us prove (P4). Arguing as before and noting that the family $\alpha^*_\varepsilon$ is increasing with respect to $\varepsilon$, we have

$$\int_{\Omega} \alpha^*_\varepsilon(\eta_{0,\varepsilon}) \leq \int_{\Omega} \alpha^*_\varepsilon(\eta_0) + \int_{\Omega} (\eta_{0,\varepsilon} - \eta_0) \alpha^{-1}_\varepsilon(\eta_{0,\varepsilon})$$

$$\leq \int_{\Omega} \alpha^*(\eta_0) + \varepsilon \int_{\Omega} \alpha^{-1}_\varepsilon(\eta_{0,\varepsilon,xx}) \eta_{0,\varepsilon,xx} \leq \int_{\Omega} \alpha^*(\eta_0),$$

where the last passage is justified by the monotonicity and Lipschitz continuity of $\alpha^{-1}_\varepsilon$. Recalling the last property in (A5), the proof of the Lemma turns out to be concluded.
4.1 Regularized problem

We shall prove the following result.

**Lemma 4.2.** Assume that (A1)-(A5) hold and that $\beta_\varepsilon$, $\alpha_\varepsilon$, and $\chi_{0,\varepsilon}$ are defined as above. Then there exists a unique quadruple of functions $(\vartheta_\varepsilon, \chi_\varepsilon, \eta_\varepsilon, \xi_\varepsilon)$ such that

\[
\begin{align*}
\vartheta_\varepsilon &\in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W), \\
\chi_\varepsilon &\in H^2(0, T; H) \cap W^{1, \infty}(0, T; V) \cap L^\infty(0, T; W), \\
\eta_\varepsilon &\in L^\infty(0, T; H), \\
\xi_\varepsilon &\in L^\infty(0, T; H),
\end{align*}
\]

and the following relations

\[
\begin{align*}
\vartheta_\varepsilon, t + \partial_\varepsilon \chi_\varepsilon, t &= \vartheta_\varepsilon, xx & \text{in } Q, \\
\eta_\varepsilon - \chi_\varepsilon, xx + \xi_\varepsilon &= \vartheta_\varepsilon - \vartheta_\varepsilon & \text{in } Q, \\
\eta_\varepsilon &\in \alpha_\varepsilon(\chi_\varepsilon, t) & \text{in } Q, \\
\xi_\varepsilon &= \beta_\varepsilon(\chi_\varepsilon) & \text{in } Q, \\
\vartheta_\varepsilon(t, 0) &= \vartheta_0 & \text{in } \Omega, \\
\chi_\varepsilon(t, 0) &= \chi_{0,\varepsilon} & \text{in } \Omega,
\end{align*}
\]

hold almost everywhere. Moreover, letting $b(t) := ||\chi_{\varepsilon, t}||_{L^1(0, t; L^\infty(\Omega))}$ for $t \in [0, T]$, we have that

\[
\min \vartheta_0 e^{-b(t)} \leq \vartheta_\varepsilon(x, t) \leq \max \vartheta_0 e^{b(t)}, \quad \forall (x, t) \in Q.
\]

The proof of the latter result is very close to the argument of [15]. However, since this setting is quite different we cannot simply refer to the above mentioned result and we need to present some details. For the sake of notational simplicity, we will drop most of the subscripts $\varepsilon$ throughout this subsection.

Let us begin with two lemmas which will be crucial in the sequel.

**Lemma 4.3.** Let (A4) hold and choose

\[
v \in L^2(0, T; L^\infty(\Omega)).
\]

Then, there exists a unique function $u \in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W)$ such that

\[
\begin{align*}
\dot{u} + vu &= u_{xx} & \text{a.e. in } Q, \\
\dot{u}(\cdot, 0) &= \vartheta_0 & \text{a.e. in } \Omega.
\end{align*}
\]

Moreover, letting $b(t) := ||v||_{L^1(0, t; L^\infty(\Omega))}$ for $t \in [0, T]$, we have that

\[
\min \vartheta_0 e^{-b(t)} \leq u(x, t) \leq \max \vartheta_0 e^{b(t)}, \quad \forall (x, t) \in Q.
\]

**Proof.** This lemma may be proved by referring to standard results on parabolic PDEs and performing, for instance, a regularization. We just give the detail of the bounds (4.26). Indeed, let us multiply (4.24) by

\[\varphi = (u - \max \vartheta_0 e^{b(t)})^+ := \max \{u - \max \vartheta_0 e^{b(t)}; 0\} \in L^2(0, T; V),\]
and integrate on $Q_t$ for $t \in (0,T]$. Indeed, we have that
\[
\int_{Q_t} \left( (\varphi + \max \vartheta_0 e^{b(s)}) \varphi^2 + \varphi_x^2 + v (\varphi + \max \vartheta_0 e^{b(s)}) \varphi \right) = 0.
\]
Hence, owing to (4.25), we readily check that
\[
\frac{1}{2} |\varphi(t)|^2 \leq - \int_{Q_t} \left( v(s) + \|v(s)\|_{L^\infty(\Omega)} \right) \max \vartheta_0 e^{b(s)} \varphi - \int_{Q_t} v \varphi^2,
\]
and, by means of sign considerations and applying the Gronwall lemma, we conclude that $\varphi = 0$ everywhere in $Q$ and the upper bound in (4.26) holds. It suffices to repeat the above argument with the choice $\varphi = -(u - \min \vartheta_0 e^{-b(t)})^-$ in order to conclude for the lower bound in (4.26). Moreover, it is a standard matter to check that both bounds in (4.26) are actually sharp. \hfill \square

**Lemma 4.4.** Let (A5), (P1) hold and let $\alpha_e$, $\beta_e$ be as in Lemma 4.2. Moreover, choose
\[ u \in H^1(0,T; H). \tag{4.27} \]
Then, there exists a unique function $v \in H^2(0,T; H) \cap W^{1,\infty}(0,T; V) \cap L^\infty(0,T; W)$ such that
\[
\alpha_e(v_t) - v_{xx} + \beta_e(v) \ni u - \vartheta_c \text{ a.e. in } Q, \tag{4.28}
\]
\[
v(x,0) = \chi_{0,e} \text{ a.e. in } \Omega. \tag{4.29}
\]

We omit the proof of the latter result since it is essentially contained in [3].

Let $R$ be a positive constant to be fixed later, and consider now the sets
\[
X(R,\tau) = \{v \in L^2(0,\tau; H) : \|v\|_{L^2(0,\tau; L^\infty(\Omega))} \leq R \}, \tag{4.30}
\]
\[
Y(R,\tau) = \{u \in H^1(0,\tau; H) : \min \vartheta_0 e^{-\sqrt{\tau} R} \leq u \leq \max \vartheta_0 e^{-\sqrt{\tau} R} \text{ a.e. in } Q_\tau \}, \tag{4.31}
\]
where $\tau \in (0,T]$ is a small time, also to be fixed later. First, we define the operator $T_1 : X(R,\tau) \to Y(R,\tau)$ mapping $v \mapsto T_1(v)$, where $T_1(v)$ stands for the unique solution (with datum $v$ and up to time $\tau$) of the problem (4.24)-(4.25). Indeed, we have that
\[
T_1(v)_t - T_1(v)_{xx} = -v \text{ a.e. in } Q_\tau. \tag{4.32}
\]
Next, we denote by $T_2 : Y(R,\tau) \to X(R,\tau)$ the operator that maps $u \mapsto T_2(u)$, where $T_2(u)$ represents the derivative in time of the unique solution (up to time $\tau$) to problem (4.28)-(4.29) where $u$ is assumed. Letting $*$ denote the standard convolution product on $(0,t)$, namely $(a * b)(t) := \int_0^t a(t-s)b(s) \, ds$, and fixed $w := \chi_{0,e} + 1 * T_2(u)$ for $t \in (0,\tau)$, one infers that
\[
\varepsilon T_2(u) + \alpha(T_2(u)) - w_{xx} + \beta_e(w) \ni u - \vartheta_c \text{ a.e. in } Q_\tau. \tag{4.33}
\]
By virtue of Lemmas 4.3 and 4.4, the operator
\[
\mathcal{T} := T_2 \circ T_1 : X(R,\tau) \to L^2(0,\tau; H),
\]
turns out to be defined. Our next aim is to prove that, for a suitably large $R$ and a correspondingly small $\tau$, the latter operator is actually a contraction on $X(R,\tau)$. First of all, we
have to check that there is a choice of $R$ and $\tau$ such that $\mathcal{T}(X(R, \tau)) \subset X(R(\tau))$. To this end, let us multiply (4.32) by $\mathcal{T}_1(v)_t$ and integrate on $Q_\tau$ in order to get

$$
\|\mathcal{T}_1(v)_t\|^2_{L^2(0,\tau;H)} \leq \frac{1}{2} \|\vartheta_0, x\|^2 + \|v\|_{L^2(0,\tau;L^\infty(\Omega))} \|\mathcal{T}_1(v)\|_{L^\infty(Q_\tau)} \|\mathcal{T}_1(v)_t\|_{L^2(0,\tau;H)}.
$$

Hence, it is straightforward to check that (recall the definition (4.31))

$$
\|\mathcal{T}_1(v)_t\|^2_{L^2(0,\tau;H)} \leq \|\vartheta_0, x\|^2 + (R \max \vartheta_0 e^{\sqrt{T\tau}})^2. \tag{4.34}
$$

Let us now multiply (4.33) by $\mathcal{T}_2(u)$ and integrate on $Q_t$ for $t \in (0, \tau)$. Owing to (A2), one readily obtains that

$$
\varepsilon \int_{Q_t} \mathcal{T}_2(u)^2 + \frac{1}{2} \|w_x(t)\|^2 + \int_{\Omega} \beta_\varepsilon(w(t)) \leq \frac{1}{2} \|\chi_{0, x, x}\|^2 + \frac{\varepsilon}{2} \int_{\Omega} \beta_\varepsilon(x_{0, x}) + \frac{1}{2\varepsilon^2} \int_{Q_t} |u - \vartheta_\varepsilon|^2.
$$

Hence, since $u \in Y(R, \tau)$, we can exploit (P3) and get that

$$
\int_{Q_t} \mathcal{T}_2(u)^2 \leq \frac{1}{\varepsilon} \|\chi_{0, x, x}\|^2 + \frac{2}{\varepsilon} \int_{\Omega} \beta_\varepsilon(x_{0, x}) + \frac{1}{\varepsilon^2} \int_{Q_t} |u - \vartheta_\varepsilon|^2
\leq \frac{1}{\varepsilon} \|\chi_{0, x, x}\|^2 + \frac{2}{\varepsilon} \left(\|\beta_\varepsilon(x_{0, x})|_{L^1(\Omega)} + r(\varepsilon)\right) + \frac{2T}{\varepsilon^2} \left(\max \vartheta_0 e^{\sqrt{T\tau}}ight)^2 + \vartheta_0^2. \tag{4.35}
$$

Next, let us take the time derivative of (4.33), multiply it by $\mathcal{T}_2(u)_t$ and integrate on $Q_t$ for $t \in (0, \tau)$. Of course, at this level this procedure is just formal since $\alpha$ need not to be differentiable. However, we could make precise this computation by introducing some ulterior approximation procedure that we prefer not to detail here for the sake of clarity. We obtain

$$
\varepsilon \int_{Q_t} \mathcal{T}_2(u)^2 + \frac{1}{2} |\mathcal{T}_2(u)_t(t)|^2 \leq \frac{1}{2} |\mathcal{T}_2(u)_t(0)|^2 + \int_{Q_t} u_t \mathcal{T}_2(u)_t + \frac{1}{\varepsilon} \left(\int_{0}^{t} |\mathcal{T}_2(u)|^2 |\mathcal{T}_2(u)_t|\right)
\leq \frac{1}{2} |\mathcal{T}_2(u)_t(0)|^2 + \frac{1}{\varepsilon^3} \int_{Q_t} \mathcal{T}_2(u)^2 + \frac{1}{\varepsilon} \left(\int_{Q_t} u_t^2 + \frac{\varepsilon}{2} \int_{Q_t} \mathcal{T}_2(u)^2. \tag{4.35}\right.
$$

In order to control the term depending on initial values, let us note that, by equation (4.33) written at the time $t = 0$ and recalling (4.5), we have

$$
\mathcal{T}_2(u)_t(0) = \alpha^{-1} \left(\vartheta_0 - \vartheta_\varepsilon + x_{0, x, x} - \xi_{0, x}\right) = \alpha^{-1} (\eta_{0, x}). \tag{4.36}
$$

Thus, owing in particular to (P1) and to the Lipschitz continuities of $\beta_\varepsilon$ and $\alpha^{-1}_\varepsilon$,

$$
|\mathcal{T}_2(u)_t(0)| \leq \frac{3}{\varepsilon^2} \left(\|\vartheta_0, x\|^2 + \|x_{0, x, x}\|_{L^2(\Omega)} + \varepsilon^{-2} \|x_{0, x}^2\right) \leq C_\varepsilon, \tag{4.37}
$$

where $C_\varepsilon$ is a computable positive constant, which of course depends on $\varepsilon$. 

Hence, it is straightforward to recall (4.34) and (4.35) and get that, for \( \varepsilon \) small enough,

\[
\int_{Q_t} \mathcal{T}_2(u)^2 + |\mathcal{T}_2(u,x(t))|^2
\leq \frac{1}{\varepsilon} C^* + \frac{2}{\varepsilon^2} \left( \frac{1}{\varepsilon} |\chi_{0,\varepsilon,x}|^2 + \frac{2}{\varepsilon} \left( \|\hat{\beta}(\chi_0)\|_{L^1(\Omega)} + r(\varepsilon) \right) + \frac{2T}{\varepsilon^2} \left( \max \vartheta_0 \, \varepsilon^{\sqrt{T R}} + \vartheta_0^2 \right) \right)
\]

\[
+ \frac{2}{\varepsilon^2} \left( |\vartheta_0|^2 + \left( R \max \vartheta_0 \, \varepsilon^{\sqrt{T R}} \right)^2 \right)
\leq \frac{1}{\varepsilon} C^* + \frac{4}{\varepsilon^2} |\chi_{0,\varepsilon,x}|^2 + \frac{4}{\varepsilon^3} \left( \|\hat{\beta}(\chi_0)\|_{L^1(\Omega)} + r(\varepsilon) \right) + \frac{2T}{\varepsilon^2} \left( \max \vartheta_0 \, \varepsilon^{\sqrt{T R}} + \vartheta_0^2 \right)
\]

\[
+ \frac{4T}{\varepsilon^6} \left( \max \vartheta_0 \, \varepsilon^{\sqrt{T R}} \right)^2 + \frac{4T \vartheta_0^2}{\varepsilon^6} \leq \frac{1}{\varepsilon^2} \left( R \max \vartheta_0 \, \varepsilon^{\sqrt{T R}} \right)^2.
\]  \tag{4.38}

Thus, we readily have that

\[
\|T(v)(t)\|_{L^\infty(\Omega)}^2 \leq \frac{C^*}{\varepsilon^2} \left( R \max \vartheta_0 \, \varepsilon^{\sqrt{T R}} \right)^2 \quad \forall t \in [0,T],
\]

where \( C^* \) depends on a suitable embedding constant and it is independent of \( \varepsilon \), and

\[
\|T(v)(t)\|_{L^2(0,t;L^\infty(\Omega))}^2 \leq \frac{C^*}{\varepsilon^2} \left( R \max \vartheta_0 \, \varepsilon^{\sqrt{T R}} \right)^2.
\]

Finally, it suffices to choose

\[
\tau \leq \varepsilon^2 \left( \sqrt{C^*} \max \vartheta_0 \, \varepsilon^{\sqrt{T R}} \right)^{-2},
\]

and it readily follows that \( T(v) \in X(R,\tau) \). We now refer to [15] for the proof of the contraction character of \( T \) in \( L^2(0,T;H) \). Let us just stress that a further constraint on the size of \( \tau \) needs to be introduced. On account of all this procedure, it suffices to recall that \( X(R,\tau) \) is closed in \( L^2(0,T;H) \) to conclude for the existence and uniqueness of a local in time solution to (4.18)-(4.21). Of course the regularities (up to time \( \tau \)) of (4.12)-(4.15) are ensured by classic results on parabolic equations. In order to conclude the proof of Lemma 4.2 it is enough to prove that the latter solution is actually global. Indeed, owing to the next subsection, it is possible to find a priori estimates for \( (\vartheta_\varepsilon, x_{\varepsilon}, \xi_{\varepsilon}, \eta_{\varepsilon}) \) independent of \( \tau \) (and \( \varepsilon \)), which allow us to extend our local unique solution to a global one, i.e. defined on the whole interval \( (0,T) \). In this concern, we prefer, for the sake of clarity, to establish the following a priori estimates directly for a global solution on \( (0,T) \) instead of carrying on with the small time parameter \( \tau \).

### 4.2 A priori estimates

Henceforth, let \( C \) denote any constant, possibly depending on \( \vartheta_0, \chi_0, C_1, C_2, C_3, \vartheta_\varepsilon, \) and \( T \) but not on \( \varepsilon \). Of course, \( C \) may vary from line to line.
**First estimate.** Let us multiply relation (4.17) by $\chi_{c,t}$, take the sum with equation (4.16), and integrate on $Q_t$ for $t \in (0,T)$. We get

$$
\int_\Omega \vartheta_c(t) + \varepsilon \int_{Q_t} \chi_{c,t}^2 + \frac{1}{2} |\chi_{c,t}(t)|^2 + \int_\Omega \hat{\beta}_c(\chi_c(t)) \\
\leq \int_\Omega \vartheta_0 + \frac{1}{2} |\chi_{0,t,x}|^2 + \int_\Omega \hat{\beta}_c(\chi_{0,x}) - \vartheta_c \int_\Omega \chi_c(t) + \vartheta_c \int_\Omega \chi_{0,x}.
$$

(4.39)

It suffices now to recall (4.3) to obtain

$$
-\vartheta_c \int_\Omega \chi_c(t) \leq \frac{C_1}{4} \int_\Omega \chi_c(t) + \frac{\vartheta_c^2}{C_1} \leq \frac{1}{2} \int_\Omega \hat{\beta}_c(\chi_c(t)) + \frac{C_2}{2} + \frac{\vartheta_c^2}{C_1}.
$$

Hence, owing to (A4)-(A5), (P3), and (4.22), we readily have that

$$
\|\vartheta_c\|_{L^\infty(0,T;L^1(\Omega))} + \|\chi_c\|_{L^\infty(0,T;V)} \leq C,
$$

(4.40)

whenever $\varepsilon$ is small enough.

**Second estimate.** Let us recall that (4.22) ensures that $1/\vartheta_c \in L^\infty(Q)$ for all $\varepsilon > 0$. Thus, we are allowed to multiply (4.16) by the function $-1/\vartheta_c$ and take the integral on $Q_t$ for $t \in (0,T)$. One has that

$$
- \int_\Omega \ln \vartheta_c(t) + \int_{Q_t} (\ln \vartheta_c)_x^2 = - \int_\Omega \ln \vartheta_0 + \int_\Omega \chi_c(t) - \int_\Omega \chi_{0,x}.
$$

Hence, also using (A4), (P1), and (4.40), we conclude for

$$
- \int_\Omega \ln \vartheta_c(t) + \int_{Q_t} (\ln \vartheta_c)_x^2 \leq C,
$$

(4.41)

for all $t \in (0,T)$. Now it is a standard matter (see, e.g., [13, Sec. 4]) to prove that we have

$$
\|\vartheta_c(t)\|_{L^\infty(\Omega)} \leq C(|\vartheta_c(t)|_{L^1(\Omega)} + \|\vartheta_c(t)\|_{L^1(\Omega)}(\ln \vartheta_c(t))_x^2\|_{L^2(\Omega)}),
$$

and (4.40)-(4.41) yield

$$
\|\vartheta_c\|_{L^1(0,T;L^\infty(\Omega))} \leq C.
$$

(4.42)

Finally, by elementary interpolation and (4.40) we conclude for

$$
\|\vartheta_c\|_{L^2(0,T;H)} \leq C.
$$

(4.43)

**Third estimate.** Let us multiply relation (4.16) by the function $(\vartheta_c + \chi_{c,t})$, multiply the derivative in time of (4.17) by $\chi_{c,t}$, sum the resulting equations, and integrate on $Q_t$ for $t \in (0,T)$. Of course, in the setting of some regularized problem, the time differentiation may be fully justified. We have

$$
\frac{1}{2} |\vartheta_c(t)|^2 + \int_{Q_t} \vartheta_c^2 + \int_{Q_t} \vartheta_c \chi_{c,t}^2 + \int_{Q_t} \eta_{c,t} \chi_{c,t} + \int_{Q_t} \chi_{c,t}^2 + \int_{Q_t} \xi_{c,t} \chi_{c,t} \leq \frac{1}{2} |\vartheta_0|^2 - \int_{Q_t} \vartheta_c^2 \chi_{c,x,t} - \int_{Q_t} \vartheta_c \chi_{c,x,t}.
$$

(4.44)
Now, it is a standard matter to perform the calculations
\[
\int_{Q_t} \eta_{\varepsilon, t} x_{\varepsilon, t} \geq \int_{\Omega} \hat{\alpha}_{\varepsilon}^*(\eta_{\varepsilon}(t)) - \int_{\Omega} \hat{\alpha}_{\varepsilon}^*(\eta_{0, \varepsilon}),
\]
\[
\int_{Q_t} \xi_{\varepsilon, t} x_{\varepsilon, t} \geq 0,
\]
\[
-\int_{Q_t} \vartheta_{\varepsilon} x_{\varepsilon, t} \leq \frac{1}{2} \int_{Q_t} \vartheta_{\varepsilon} x_{\varepsilon, t}^2 + \frac{1}{2} \int_{Q_t} \vartheta_{\varepsilon}^3,
\]
\[
-\int_{Q_t} \vartheta_{\varepsilon, x} x_{\varepsilon, x} \leq \frac{1}{2} \int_{Q_t} \vartheta_{\varepsilon, x}^2 + \frac{1}{2} \int_{Q_t} \chi_{\varepsilon, x}^2.
\]
Moreover, we exploit (4.40) in order to get
\[
\int_{Q_t} \vartheta_{\varepsilon}^2 \leq \|\vartheta_{\varepsilon}\|_{L^\infty(0, T; L^1(\Omega))} \|\vartheta_{\varepsilon}^2\|_{L^1(0, T; L^\infty(\Omega))} \leq C\|\vartheta_{\varepsilon}\|_{L^2(0, T; L^\infty(\Omega))}^2,
\]
and the compactness of the injection $V \subset L^\infty(\Omega)$ ensures that, for any fixed $\delta > 0$, there exists a positive constant $C_\delta$ such that
\[
\|u\|_{L^\infty(\Omega)} \leq \delta |u_x| + C_\delta |u| \quad \forall u \in V.
\]
Hence, we readily prove that
\[
\int_{Q_t} \vartheta_{\varepsilon}^2 \leq C \int_{Q_t} (\delta \vartheta_{\varepsilon}^2 + C_\delta \vartheta_{\varepsilon}^2) \leq C\delta \int_{Q_t} \vartheta_{\varepsilon, x}^2 + CC_\delta \int_{Q_t} \vartheta_{\varepsilon}^2,
\]
with obvious notation. Finally, taking into account (A4) and (P4) and using again Gronwall’s lemma, we conclude for
\[
\|\vartheta_{\varepsilon}\|_{L^\infty(0, T; H) \cap L^2(0, T; V)} + \|\chi_{\varepsilon, x}\|_{L^2(0, T; H)} \leq C,
\]
(4.45)
\[
\int_{Q_t} \vartheta_{\varepsilon} x_{\varepsilon, t} \leq C.
\]
(4.46)

**Fourth estimate.** Let $t \in (0, T)$, $\gamma \in (0, 1)$ and define the set
\[
\omega_{\varepsilon}(t) := \left\{ x \in \Omega : \vartheta_{\varepsilon}(x, t) \geq \gamma \right\},
\]
(recall (4.12)). Owing to (4.41), it a standard matter to find a $\gamma \in (0, 1)$ such that
\[
|\omega_{\varepsilon}(t)| \geq \frac{1}{2} \quad \forall \varepsilon > 0 \quad \text{and a.e. } t \in (0, T),
\]
where $| \cdot |$ stands for the Lebesgue measure. Indeed, denoting by $\omega_{\varepsilon}^c(t)$ the complement of $\omega_{\varepsilon}(t)$, it suffices to observe that
\[
-C \leq \int_{\Omega} \ln \vartheta_{\varepsilon}(t) = \int_{\omega_{\varepsilon}(t)} \ln \vartheta_{\varepsilon}(t) + \int_{\omega_{\varepsilon}^c(t)} \ln \vartheta_{\varepsilon}(t)
\]
\[
\leq \int_{\Omega} \vartheta_{\varepsilon}(t) + (\ln \gamma) |\omega_{\varepsilon}^c(t)|,
\]
uniformly with respect to $\varepsilon$ and almost everywhere in time. In particular, we have that $\gamma \to 0$ implies $|\omega_{\varepsilon}^c(t)| \to 0$. Hence, whenever $\gamma$ goes to 0, the measure $|\omega_{\varepsilon}(t)|$ approaches 1 uniformly in $\varepsilon$ and almost everywhere in $(0, T)$. 

Then, it is easy to compute that (recall now (4.13))
\[
\chi^2_{\varepsilon,t}(x,t) \leq \frac{2}{|\omega_\varepsilon(t)|} \int_{\omega_\varepsilon(t)} \chi^2_{\varepsilon,t}(t) + 2 \int_{\Omega} \chi^2_{\varepsilon,xt}(t) \quad \text{for a.e. } (x,t) \in Q,
\]
which entails, according to the above argument,
\[
\int_{Q} \chi^2_{\varepsilon,t} \leq \frac{4}{\tau} \int_{Q} \partial_{\varepsilon} \chi^2_{\varepsilon,t} + 2T \int_{Q} \chi^2_{\varepsilon,xt}.
\]
Finally, estimate (4.45) allows us to conclude that
\[
\|\chi_{\varepsilon,t}\|_{L^2(0,T;V)} \leq C. \tag{4.47}
\]

Fifth estimate. Taking into account (4.45), (4.47), and recalling the continuity of the injection \(V \subset L^\infty(\Omega)\), we readily get that the term \(\partial_{\varepsilon} \chi_{\varepsilon,t}\) is bounded in \(L^2(0,T;H)\) uniformly with respect to \(\varepsilon\). Hence it is a standard matter to deduce from (4.16) and (A4) the usual parabolic estimates
\[
\|\partial_{\varepsilon}\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \leq C. \tag{4.48}
\]

Sixth estimate. We now multiply relation (4.17) by the function \((-\chi_{\varepsilon,xx} + \xi_\varepsilon)_t\) and take the integral on \(Q_t\) for \(t \in (0,T)\). Let us stress that the latter procedure is, at this stage, just formal. Indeed the regularities stated in (4.13) and (4.15) do not allow us to take time derivatives. Nevertheless we claim that the above mentioned argument can be made precise and we refer the reader to [15, Remark 3.5] for further details. We obtain
\[
\frac{1}{2} |(-\chi_{\varepsilon,xx} + \xi_\varepsilon)(t)|^2 - \int_{Q_t} \eta_\varepsilon \chi_{\varepsilon,xx} + \int_{Q_t} \eta_\varepsilon \xi_\varepsilon,
\]
\[
= \frac{1}{2} |(-\chi_{0,xx} + \xi_0)_t|^2 + \int_{Q_t} (\partial_{\varepsilon} - \partial_{\varepsilon})(-\chi_{\varepsilon,xx} + \xi_\varepsilon)_t, \tag{4.49}
\]
and the last term in the right hand side above may be controlled by means of (A4) and (P2) as follows
\[
\int_{Q_t} (\partial_{\varepsilon} - \partial_{\varepsilon})(-\chi_{\varepsilon,xx} + \xi_\varepsilon)_t,
\]
\[
= \int_{\Omega} (\partial_{\varepsilon} - \partial_{\varepsilon})(-\chi_{0,xx} + \xi_0)_t - \int_{\Omega} (\partial_0 - \partial_{\varepsilon})(-\chi_{0,xx} + \xi_0) - \int_{Q_t} \partial_{\varepsilon,t}(-\chi_{\varepsilon,xx} + \xi_\varepsilon)
\]
\[
\leq \frac{1}{4} |(-\chi_{0,xx} + \xi_0)(t)|^2 + 2|\partial_{\varepsilon}(t)|^2 + \int_0^t |\partial_{\varepsilon,t}| | - \chi_{\varepsilon,xx} + \xi_\varepsilon | + C.
\]
As for the left hand side of (4.49) we readily check that, by monotonicity of \(\alpha_\varepsilon\) and \(\alpha_\varepsilon(0) = 0\),
\[
\int_{Q_t} \eta_\varepsilon \xi_{\varepsilon,t} = \int_{Q_t} \eta_\varepsilon \beta_\varepsilon(x_\varepsilon) \chi_{\varepsilon,t} \geq 0.
\]
On the other hand it is clear that
\[
- \int_{Q_t} \eta_\varepsilon \chi_{\varepsilon,xx} \geq 0
\]
according to the monotonicity of $\alpha_\varepsilon$ and [7, Lemma 2]. Finally, taking into account (A4) and (4.48), we apply the Gronwall lemma and, using again the monotonicity of $\beta_\varepsilon$, we obtain

$$\|x_{\varepsilon;x}\|_{L^\infty(0,T;H)} + \|\xi_\varepsilon\|_{L^\infty(0,T;H)} \leq C. \quad (4.50)$$

**Seventh estimate.** A comparison in (4.17) and estimates (4.45) and (4.50) entail

$$\|\eta_\varepsilon\|_{L^\infty(0,T;H)} \leq C. \quad (4.51)$$

### 4.3 Passage to the limit

In order to pass to the limit in (4.16)-(4.19) we start by observing that the two operator convergences

$$\alpha_\varepsilon \to \alpha, \quad \beta_\varepsilon \to \beta \quad (4.52)$$

hold in the sense of the $G$-convergence of graphs in $\mathbb{R} \times \mathbb{R}$. Namely, for all $(x,y) \in \mathbb{R} \times \mathbb{R}$ such that $y \in \alpha(x)$ ($\beta(x)$, respectively) there exists a sequence $(x_\varepsilon, y_\varepsilon) \in \mathbb{R} \times \mathbb{R}$ such that $y_\varepsilon \in \alpha_\varepsilon(x_\varepsilon)$ ($\beta_\varepsilon(x_\varepsilon)$, respectively) and $(x_\varepsilon, y_\varepsilon)$ converges to $(x,y)$ as $\varepsilon \to 0$. The latter convergences turn out to be crucial in the limit procedure.

Owing to (4.47)-(4.48), (4.50)-(4.51), and to well-known compactness results, we may find a quadruple $(\vartheta, \chi, \eta, \xi)$ such that, possibly taking subsequences (not relabeled), one has that

$$\vartheta_\varepsilon \to \vartheta \quad \text{weakly in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \quad \text{and strongly in } C([0,T];H) \cap L^2(0,T;V), \quad (4.53)$$

$$\chi_\varepsilon \to \chi \quad \text{weakly in } H^1(0,T;V) \cap L^\infty(0,T;W) \quad \text{and strongly in } C([0,T];V), \quad (4.54)$$

$$\eta_\varepsilon \to \eta \quad \text{weakly in } L^\infty(0,T;H), \quad (4.55)$$

$$\xi_\varepsilon \to \xi \quad \text{weakly in } L^\infty(0,T;H). \quad (4.56)$$

It is thus possible to pass to the limit in (4.16)-(4.17) obtaining (3.6)-(3.7), respectively, along with the properties (3.2)-(3.5). Of course, also because of (P2), the initial conditions (3.10)-(3.11) are fulfilled.

Moreover, the convergences (4.54) and (4.56) entail

$$\lim_{\varepsilon \to 0} \int_Q \xi_\varepsilon \chi_\varepsilon = \int_Q \xi \chi.$$

Thus, relation (3.9) is an easy consequence of the second operator convergence in (4.52) and classical results on monotone operators (see, e.g., [2, Prop. 1.1.iv, p. 42]).

Let us now multiply equation (4.17) by $x_{\varepsilon;t}$ and integrate over $Q$ obtaining

$$\int_Q \eta_\varepsilon x_{\varepsilon;t} = -\int_Q x_{\varepsilon;x} x_{\varepsilon;x} - \int_Q \xi_\varepsilon x_{\varepsilon:t} + \int_Q (\vartheta_\varepsilon - \vartheta) x_{\varepsilon;t}. \quad (4.57)$$

Our next aim is that of passing to the lim sup as $\varepsilon \to 0$ in the above equation. It is well known that, thanks to the convergence (4.52), the functionals induced by $\beta_\varepsilon$ on $H$, namely

$$B_\varepsilon(v) := \int_{\Omega} \beta_\varepsilon(v(x)) \, dx \quad \text{for } v \in H, \quad (4.58)$$
turn out to converge in the sense of Mosco [1, Prop. 3.56, p. 354] in $H$ to the functional

$$B(v) := \begin{cases} \int_{\Omega} \hat{\beta}(v(x)) \, dx & \text{if } v \in H \text{ and } \hat{\beta}(v) \in L^1(\Omega) \\
+\infty & \text{if } v \in H \text{ and } \hat{\beta}(v) \not\in L^1(\Omega). \end{cases} \quad (4.59)$$

In particular, owing to (4.54), one has that

$$\int_{\Omega} \hat{\beta}(x(T)) \leq \liminf_{\varepsilon \to 0} \int_{\Omega} \hat{\beta}_\varepsilon(x(T)).$$

Hence, using (3.9), (4.2), and (P3), one readily deduces that

$$\limsup_{\varepsilon \to 0} \left( -\int_{Q} \xi \chi_{\varepsilon,t} \right) = -\liminf_{\varepsilon \to 0} \int_{\Omega} (\hat{\beta}_\varepsilon(x(T)) - \hat{\beta}_\varepsilon(x_{0,\varepsilon})) \leq -\int_{\Omega} (\hat{\beta}(x(T)) - \hat{\beta}(x_0)) = -\int_{Q} \xi \chi_t. \quad (4.60)$$

Then, thanks also to (4.53)-(4.54), taking the lim sup of (4.57) yields

$$\limsup_{\varepsilon \to 0} \int_{Q} \eta \chi_{\varepsilon,t} \leq -\int_{Q} \chi_{x,x_{xt}} - \int_{Q} \xi \chi_t + \int_{Q} (\theta - \theta_c) \chi_t = \int_{Q} \eta \chi_t,$$

where the last equality holds since (3.7) is satisfied in the limit. According to the latter relation, the inclusion (3.8) is again a standard consequence of the convergences (4.52), (4.54)-(4.55), and [2, Lemma 1.3, p. 42]. Finally, the bounds (3.12) can be checked by simply reproducing the argument of Lemma 4.3.

References


