Materials with Elastic Range and the Possibility of Stress Oscillations in Pure Shear

M. Lucchesi
CNUCE-CNR

P. Podio-Guidugli
Universita' di Roma 2

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M.Lucchesi
Istituto CNUCE - C.N.R.
Via S.Maria, 36
56100 Pisa, Italy

P.Podio-Guidugli
Dipartimento di Ingegneria Civile Edile
Università di Roma 2
00173 Roma, Italy

SUMMARY - The framework of a general theory of materials with elastic range [6] is used to show that, for an isotropic material undergoing a process in which the difference $(D - D^p)$ between the total and the plastic rate of deformation is uniformly small, the incremental constitutive equation features an isotropic dependence of the Jaumann derivative of the Cauchy stress T on $(D - D^p)$. Thus, it is argued that stress oscillations in pure shear should be seen as a (not desired) consequence of the choice made for the hardening rule rather than, as was also conjectured, of the choice made for the invariant time derivative of T to appear in the constitutive equation.

1. INTRODUCTION

In 1981 Nagtegaal & De Jong [1] studied how the Cauchy stress T in an elastic-plastic material evolves during a large deformation of pure shearing. Sticking to current practice, they assumed that the Jaumann derivative of T, namely,

$$(1.1) \qquad \partial_J T := \dot{T} + TW - WT ,$$

with W the spin tensor, depended both linearly and isotropically on the difference between the total rate of deformation D and the plastic rate of deformation D^p :

$$(1.2) \qquad \partial_J T = 2\mu(D - D^p) + \lambda\, tr(D - D^p)I \ .$$

Moreover, in order to take Bauschinger effect into account, they also assumed that the Jaumann derivative of the "back stress" C [(1)] were proportional to the plastic rate of deformation:

$$(1.3) \qquad \partial_J C = 2\gamma D^p ,$$

applying the classical kinematic hardening rule, originally proposed by Melan [2] for small deformations only. Under those hypotheses, Nagtegaal & De Jong found that stress oscillations are in order, which are both unexpected from the physical point of view and in contrast with the available experimental evidences.

To remove this difficulty, various proposals have been advanced (*cf. e.g.* [3] and [4]) to the effect of replacing the Jaumann derivative in the constitutive prescriptions (1.2) and (1.3) by one or another objective operation of differentiation with respect to time, *i.e.* , replacing the spin W in definition (1.1) by some other suitable skew-symmetric tensor. Recently, Reed & Atluri [5] have questioned such proposals, for two good reasons. Firstly, they have remarked that all attempts to replace the Jaumann derivative with something else have failed to yield the desired results, especially so in the case of pure shear deformations. Secondly, they have shown that stress oscillations do disappear as soon as in (1.3) the Jaumann derivative of the back stress is allowed to depend on the deformation history not simply through the current plastic rate of deformation.

In this paper, supplementing the points made by Reed & Atluri, we make a third one: under reasonable hypotheses on both the material class and the process class, (1.2) is a direct outcome of a fairly general theory of elastic-plastic behavior recently proposed by Lucchesi & Podio-Guidugli [6] as an application-oriented reformulation of former theories of Lee & Liu [7], Owen [8] and /ilhavý [9]. In particular, the occurrence of the Jaumann derivative in (1.2) is automatic when one

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(1) In plasticity theories for which these notions make sense, the back stress is the center of the yield surface in the stress space.

restricts attention to those process classes, of interest in many applications, for which the difference $(D - D^p)$ is, in a sense we make precise, small.

2. NOTATIONS AND PRELIMINARIES

Once and for all we refer the reader to [6] for a detailed exposition of the main features of the theory of materials with elastic range. However, at the expenses of reducing substantially the generality of certain concepts and developments, we strive hereafter to make the present paper reasonably self-contained. We begin by introducing some notation, quoting from [6] almost *verbatim*.

For $\mathbf{V}$ a three-dimensional vector space, Lin denotes the collection of all linear transformations (second-order tensors) on $\mathbf{V}$, with I the identical transformation. Subcollections of Lin to be considered here are:

Lin$^+$ = all elements of Lin with positive determinant;

Skw = all skew-symmetric elements of Lin;

Sym = all symmetric elements of Lin;

Sym$^+$= all positive-definite elements of Sym;

Rot = all rotations of $\mathbf{V}$, *i.e.*, all orthogonal elements of Lin$^+$.

Lin can be made into an inner-product space by defining, for all $A,B \in$ Lin,

$$A \cdot B := \mathrm{tr}(AB^T)\,, \tag{2.1}$$

with tr the trace functional, and B^T the transpose of B. Lin shares with its subspaces the metric induced by this inner product; we write

$$\|A\| := (A \cdot A)^{1/2} \tag{2.2}$$

for the modulus of A.

For δ a positive real number, a history of duration δ is a continuous and piecewise continuously differentiable mapping

$$\hat{F} : [0, \delta] \rightarrow \mathrm{Lin}^+\,, \quad F = \hat{F}(\tau) \tag{2.3}$$

delivering the deformation gradient F at time τ (at a fixed material point, with respect to a fixed reference configuration); in particular, a history is rigid if the mapping (2.3) takes its values in Rot. The product of two histories $\hat{F}$ and $\hat{G}$ is the history $\hat{F}\hat{G}(\tau) := \hat{F}(\tau)\hat{G}(\tau)$ for all $\tau \in [0, \delta]$. We denote a rigid history with constant value Q by $Q^{\dagger}$, and by $\hat{F}Q^{\dagger}$ the product history of $\hat{F}$ and $Q^{\dagger}$.

Roughly speaking, the materials to be considered here are elastic-plastic isotropic solids whose mechanical response to deformation processes is described by an objective and rate-independent constitutive functional.

Let $\mathcal{D}$ the collection of all histories $\hat{F}$ whose initial value $\hat{F}(0)$ is a rotation, and let $\hat{F} \in \mathcal{D}$ and $\tau \in [0,\delta]$ be arbitrarily fixed. We denote by $\hat{T}_F(\tau)$ the Cauchy stress at time τ associated with history $\hat{F}$ by the constitutive functional; as the material is assumed to be isotropic, for all $Q \in$ Rot and for all histories $\hat{F}Q^{\dagger}$ we have that

$$(2.4) \qquad \hat{T}_{FQ}(\tau) = \hat{T}_F(\tau) \ .$$

The type of elastic-plastic behavior we have in mind is further specified by introducing the notions of elastic range and unloading history.

The elastic range at time τ corresponding to the history $\hat{F}$ is a non-empty set $\mathcal{E}_F(\tau) \subset \mathrm{Lin}^+$, whose points are interpreted as gradients of deformations from the reference configuration to configurations which are elastically accessible from the current configuration itself.

We assume that $\hat{F}$ can be multiplicatively decomposed into an elastic history $\hat{E}_F$ and an unloading history $\hat{S}_F$:

$$(2.5) \qquad \hat{F}(\tau) = \hat{E}_F(\tau)\hat{S}_F(\tau) \ ,$$

with $\hat{S}_F(\tau) \in \mathcal{E}_F(\tau)$ corresponding to a configuration in which the stress is null. We also assume, as is appropriate for solids, that there exists a unique positive-definite unloading history $\hat{S}_F^{\ +}$, so that, in view also of the assumption of objectivity, the collection $\mathcal{S}(\hat{F})$ of all unloading histories associated with $\hat{F}$ can be represented as

$$\mathcal{S}(\hat{F}) = \{\hat{Q}\hat{S}_F{}^+ \,|\, \hat{Q} = \text{a rigid history}, \ \hat{S}_F{}^+(\tau) \in \mathrm{Sym}^+\} .$$

As is shown in [6], it follows from the constitutive hypotheses just listed that the mechanical response is fully described by the structural mapping T^*, *i.e.*, an objective and isotropic mapping from a neighbourhood in Lin^+ of the identity tensor I into Sym, such that

$$(2.6) \qquad \hat{T}_F(\tau) = T^*\big(\hat{F}(\tau)(\hat{S}_F(\tau))^{-1}\big) \ \text{ for all } \hat{S}_F \in \mathcal{S}(\hat{F})$$

and

$$(2.7) \qquad T^*(Q) = 0 \ \text{ for all } Q \in \mathrm{Rot} .$$

We assume here that there is no plastic change of volume, *i.e.*, that

$$(2.8) \qquad \det \hat{S}_F(\tau) = 1 \ \text{ for all } \hat{S}_F \in \mathcal{S}(\hat{F}) ;$$

in particular,

$$(2.9) \qquad \det \hat{S}_F{}^+(0) = 1 .$$

But, as the material is isotropic, $\hat{S}_F{}^+(0)$ must be a multiple of the identity tensor I (*c.f.* [6]). Therefore, (2.9) implies that, for all $\hat{F} \in \mathcal{D}$ and for all $\hat{S}_F \in \mathcal{S}(\hat{F})$, the initial value $\hat{S}_F(0)$ of $\hat{S}_F$ must be a rotation. Moreover, as $\hat{F}(0)(\hat{S}_F(0))^{-1} \in \mathrm{Rot}$, (2.6) and (2.7) imply that the reference configuration is stress-free:

$$(2.10) \qquad \hat{T}_F(0) = 0 .$$

From now on, in (2.6) we shall systematically select that unloading history $\hat{P}_F$ whose associated elastic history

(2.11) $$\hat{V}_F := \hat{F}(\hat{P}_F)^{-1}$$

is symmetric and positive-definite (so that, in particular, as $\hat{V}_F(0) \in Sym^+ \cap Rot$,

(2.12) $$\hat{V}_F(0) = I \ ).$$

Accordingly, we shall write (2.6) as

(2.13) $$\hat{T}_F(\tau) = T^*(\hat{V}_F(\tau)) .$$

Moreover, we shall denote by $L := \dot{F}F^{-1}$ the spatial gradient of velocity, whose symmetric and skew-symmetric parts $D := ½(L + L^T)$ and $W := ½(L - L^T)$ are, respectively, the rate of deformation and the spin associated with the history $\hat{F}$ (notice that here and henceforth a superposed dot indicates time differentiation; *e.g.*, $\dot{F}$ stands for the value, at time τ, of the time derivative of $\hat{F}$). Similar notation and terminology we shall use for the corresponding constructs based on $\hat{P}_F$, namely, the plastic gradient of velocity $L^p := \dot{P}P^{-1}$, rate of deformation D^p and spin W^p.

3. THE CONSTITUTIVE LAW FOR THE STRESS RATE

In a number of applications, the difference $(D - D^p)$ between the total and the plastic rate of deformation remains small in all deformation processes of interest. To formalize such a situation, for $\varepsilon > 0$ fixed, we denote by $\mathcal{D}_\varepsilon$ the subset of all histories $\hat{F} \in \mathcal{D}$ such that

(3.1) $\qquad \| D(\tau) - D^p(\tau) \| \le \varepsilon/\delta$ for all $\tau \in [0,\delta]$.

As a preparatory result, we begin by showing that $\| \hat{V}_F(\tau) - I \|$ is small whenever $\| D(\tau) - D^p(\tau) \|$ is.

<u>LEMMA 1</u> For all $\hat{F} \in \mathfrak{D}_\varepsilon$ and for all $\tau \in [0,\delta]$,

$$\| \hat{V}_F(\tau) - I \| \le \exp(\varepsilon\tau/\delta) - 1 \; .$$

<u>Proof</u> Set $H := V - I$, and consider the mapping $\tau \mapsto H(\tau)$. In view of (2.12), $H(0) = 0$; as the lemma is trivially true if $H(\tau) \equiv 0$ over $[0,\delta]$, we henceforth assume that this is not the case. Taking into account (2.11), we have that

(3.2) $$\dot{H} = \dot{V} = \dot{F}P^{-1} - FP^{-1}\dot{P}P^{-1} = LV - VL^p = L - L^p + LH - HL^p \; .$$

As $V \in \mathrm{Sym}$, we deduce from (3.2) that

(3.3) $$\dot{H} \cdot H = (D - D^p) \cdot H + (D - D^p) \cdot H^2 \; ;$$

consequently, by applying some elementary algebraic inequalities, we have that

(3.4) $$\dot{H} \cdot H \le \| D - D^p \| \, (1 + \| H \|) \| H \| \; .$$

For $\tau \in [0, \delta]$ fixed, let $\| H(\tau^\circ) \|$ be the maximum value of $\tau' \mapsto \| H(\tau') \|$ over $[0,\tau]$, and let $\| H(\tau^\circ) \| > 0$, the non-trivial case. As the norm mapping is smooth away from zero, and as we have already remarked that $H(0) = 0$, there exists $\tau_\circ \in [0,\tau^\circ[$ such that $\| H(\tau_\circ) \| = 0$ and $\| H(\tau') \| > 0$ for all $\tau' \in]\tau_\circ,\tau^\circ]$. Over $]\tau_\circ,\tau^\circ]$, it makes sense to write

(3.5) $$\dot{H} \cdot H = \| H \|^{\cdot} \, \| H \| \; ,$$

and we have from (3.1), (3.4) and (3.5) that

(3.6) $$(1 + \|H\|)^{-1}\|H\|^{\cdot} = \left(\ln(1 + \|H\|)\right)^{\cdot} \leq \varepsilon/\delta .$$

By integration, $(3.6)_2$ yields

$$\ln(1 + \|H(\tau^{\circ})\| \leq \varepsilon(\tau^{\circ} - \tau_{\circ})/\delta \leq \varepsilon\tau/\delta ;$$

this in turn implies

$$\|H(\tau)\| \leq \|H(\tau^{\circ})\| \leq \exp(\varepsilon\tau/\delta) - 1,$$

i.e., the desired conclusion. □

As (3.3) shows at a glance, both H and $\dot{H}$ must be small in order that ($D - D^P$) be small. In other words, (3.1) could not in any way be regarded as a constitutive property, perhaps to be guaranteed by a direct hypothesis on the domain $\mathfrak{D}$ of the constitutive functional, such as, say, that the material can suffer small elastic deformations only, so that H is small for all possible deformation processes.

Now, for $\hat{M}$ a mapping from Lin^+ into Sym, recall that $\hat{M}$ is said to be:
(i) objective, if

(3.7) $$\hat{M}(F) = Q^T\hat{M}(QF)Q ;$$

(ii) isotropic, if

(3.8) $$\hat{M}(F) = \hat{M}(FQ) ,$$

identically for $F \in Lin^+$ and $Q \in Rot$. Furthermore, recall that the "elasticity tensor" associated with $\hat{M}$ at F is the linear transformation of Lin into Sym defined by

(3.9) $$D\hat{M}(F)[A] := \lim_{\varepsilon \to 0} \varepsilon^{-1}\left(\hat{M}(F + \varepsilon A) - \hat{M}(F)\right) .$$

In the next preparatory lemma we collect without proof those properties of elasticity tensors associated with objective or isotropic mappings which are relevant to our later developments.

LEMMA 2 Let $\hat{M}$ be a smooth mapping from Lin^+ into Sym. Then, for F in the domain of $\hat{M}$ and for all $W \in Skw$,

(i) if $\hat{M}$ is objective,

$$D\hat{M}(F)[WF] = W\hat{M}(F) - \hat{M}(F)W \; ; \tag{3.10}$$

(ii) if $\hat{M}$ is isotropic,

$$D\hat{M}(F)[FW] = 0 \; . \tag{3.11}$$

Moreover, if $\hat{M}$ is objective and isotropic, there exist two constants α_M and β_M such that the following representation formula holds for $D\hat{M}(I)$:

$$D\hat{M}(I)[A] = \alpha_M A + \beta_M (trA)I \quad \text{for all } A \in Sym. \tag{3.12}$$

We are now in a position to state and prove the main result of this paper.

PROPOSITION Consider the materials with elastic range described in [6] and the preceding section. Then, for all $\hat{F} \in \mathfrak{D}_\varepsilon$, there exist two material constants λ and μ, the so-called Lamé moduli, such that

$$\partial_J T = 2\mu(D - D^p) + \lambda tr(D - D^p)I + o(\varepsilon) \; . \tag{3.13}$$

Proof Combining (2.11) and (2.13) we have that

$$T = T^*(V) = T^*(FP^{-1}) \; . \tag{3.14}$$

Differentiating (3.14) with respect to time, and taking $(3.2)_3$ into account, we get

$$(3.15) \qquad \dot{T} = DT^*(V)[\dot{V}] = DT^*(V)[LV - VL^p] \,,$$

or rather, recalling that $L = D + W$ and exploiting the linearity of $DT^*(V)$,

$$(3.16) \qquad \dot{T} = DT^*(V)[WV] + DT^*(V)[DV - VD^p] - DT^*(V)[VW^p] \,.$$

Perusal of (3.10) and (3.11) yields:

$$DT^*(V)[WV] = WT - TW \;; \quad DT^*(V)[VW^p] = 0 \,.$$

Thus, we can give to (3.16) the provisional form

$$(3.17) \qquad \dot{T} + TW - WT = DT^*(V)[DV - VD^p] \,.$$

For $A, B \in Lin$ fixed, consider now the quadratic mapping

$$V \mapsto DT^*(V)[AV - VB]$$

of Sym into itself. It is not difficult to see that, for $V = I + H$,

$$(3.18) \qquad DT^*(V)[AV - VB] = DT^*(I)[A - B] + $$
$$+ D^2T^*(I)[H][A - B] + DT^*(I)[(A - B)H] + o(H) \,,$$

where

$$DT^*(I)[(A - B)H] = DT^*(I)[AH - HB]$$

in view again of (3.11).
For all $\hat{F} \in \mathcal{D}_\varepsilon$, on recalling Lemma 1, we then have from (3.18) that

(3.19) $$DT^*(V)[DV - VD^p] = DT^*(I)[D - D^p] + o(\varepsilon) .$$

Thus, in conclusion, (3.13) is obtained from (3.17) and (3.19) by calling upon the representation formulae (1.1) and (3.12). □

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