ON THE SEQUENTIAL DIAGNOSIBILITY OF A CLASS OF DIGITAL SYSTEMS

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Abstract

A new technique for determining lower bounds to the value of sequential diagnosibility of a digital system is developed. Such technique is extended to a probabilistic model of digital systems.

(Extended Abstract)

1. Introduction

We study in this paper a problem concerning diagnosis of digital systems. We use a model that was first introduced by Preparata, Metze, and Chien[1]. In this model, a digital system is partitioned into a certain number of units, each of which can be at one of two possible states, fault-free ($\bar{F}$) and faulty (F). A configuration of a system is an assignment of either the fault-free or the faulty state to each unit in the system. We assume that each unit in the system possesses a certain amount of computational resources to enable it to test one or more of the other units in the system. The outcome of a test is a binary signal which depends on the state of the testing and the tested units. In particular, we assume that:

(i) if a fault-free unit is tested by a fault-free unit, a signal 0 will be generated,

(ii) if a faulty unit is tested by a fault-free unit, a signal 1 will be generated,

(iii) if a fault-free or a faulty unit is tested by a faulty unit, either a signal 0 or a signal 1 will be generated. (In other words, the signal generated by a faulty testing unit is completely unreliable.)

A diagnosis experiment is one in which every unit tests all the units it is capable of testing once. The outcomes of the tests are referred to as a syndrome.

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In graph theoretic terms, a digital system can be described by a directed graph G = (V, E) where the vertices represent the units of the system. An edge (v_i, v_j) in G indicates that the unit v_i is capable of testing unit v_j. A configuration is an assignment of the values $\bar{F}$ and F to the vertices in V. A syndrome is an assignment of the values 0 and 1 to the edges in E. A syndrome is said to be consistent with a configuration if conditions (i)(ii)(iii) above are not violated. We note that a given syndrome might be consistent with a number of different configurations. (However, because of (iii) in our assumption above, any syndrome is consistent with at least one configuration, namely, the configuration in which all units are faulty.)

The goal of a diagnosis experiment is to identify one or more of the faulty units in the system. A one-step diagnosis is one in which all faulty units in the system are identified. A sequential diagnosis is one in which at least one faulty unit, if there is any, is identified. For any system, both one-step diagnosis and sequential diagnosis are possible, provided that the number of faulty units does not exceed certain critical value. The one-step diagnosibility of a digital system, t_0, is defined as follows: For any given syndrome S, we can unambiguously identify all the faulty units in the system provided it is known that the system contains no more than t_0 faulty units. In other words, among all the configurations that S is consistent with, atmost one of them contains no more than t_0 faulty units. The sequential diagnosibility of a digital system, t_r, is defined as follows: For any given syndrome S, we can unambiguously identify at least one of the faulty units in the system, if there is any, provided it is known that the system contains no more than t_r faulty units. In other words, among all the configurations that S is consistent with, let $\{C_1, C_2, \ldots, C_i\}$ denote the set of configurations each of which contains no more than t_r faulty units. Then, either this set is empty (there is no configuration that S is consistent with and contains no more than t_r faulty units) or there is a unit that is faulty in all the configurations $C_1, C_2, \ldots, C_i$. In graph theoretic

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terms both t_0 and t_r are invariants of the graph $G = (V, E)$. The problem of determining t_0 and t_r is, in general, a difficult one[2]. In this paper, we show some useful techniques for obtaining a lower bound on the value of t_r. This lower bound is tight for a class of digital systems. Furthermore, our results can be extended readily to a probabilistic model of system diagnosis.

2. A General Result

Throughout our discussion, we shall assume G to be a strongly connected graph. In this case, for any given syndrome sequential diagnosis is possible if we can unambiguously identify a unit to be faulty or to be fault-free. (Clearly, our goal is achieved if a unit is identified as faulty. On the other hand, if a unit is identified as fault-free then any unit tested by this unit will be fault-free if a 0 signal results and any unit tested by this unit will be faulty if a 1 signal results. Repeating such an argument if necessary, because G is strongly connected, either a faulty unit is identified eventually or all units in the system are confirmed to be fault-free.)

For a given syndrome S, for a vertex v in G, we use $G_0^S(v)$ to denote the minimum number of faulty units in the configuration(s) that S is consistent with in which v is fault-free. Also, we use $G_1^S(v)$ to denote the minimum number of faulty units in the configuration(s) that S is consistent with in which v is faulty. If

$$t_r \leq \max(G_0^S(v),\ G_1^S(v)) - 1$$

then v can be identified unambiguously as faulty if $G_0^S(v) > G_1^S(v)$, and as fault-free if $G_0^S < G_1^S(v)$.* Consequently, the sequential diagnosibility of a graph G can be computed as

$$t_r = \min_S \left[\max_{v \in V} \left[\max(G_0^S(v),\ G_1^S(v)) \right] \right] - 1 \qquad (1)$$

In general, it is still very tedious to compute t_r using Eq. (1). Rather, we shall establish lower bounds on t_r based on Eq. (1).

A directed graph T is called a <u>2-star</u> if

(i) T is a rooted tree with all edges directed toward root v.

(ii) With the exception of v, all internal nodes have indegree 1.

(iii) The height of T is atmost 2.

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* If $G_0^S(v) = G_1^S(v)$ then all configurations that S is consistent with contains more than t_r faulty units.

Figure 1 shows an example of a 2-star. The size of a 2-star T is defined to be the number of vertices in T. For example, the size of the 2-star in Figure 1 is 9.

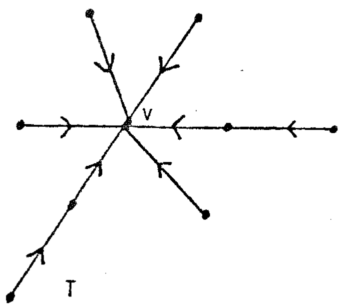

Figure 1

We state our major result without proof:

<u>Theorem 1</u>: Let G be a directed graph that contains a 2-star of size k, then

$$t_r \geq \left\lceil \frac{k}{2} \right\rceil - 1$$

As an immediate application of Theorem 1, we note that for the graph H shown in Figure 2(a), because H contains a 2-star as shown in Figure 2(b), we must have

$$t_r \geq \left\lceil \frac{2p+1}{2} \right\rceil - 1 = p$$

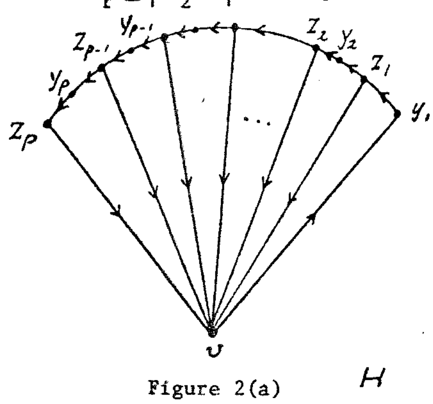

Figure 2(a)

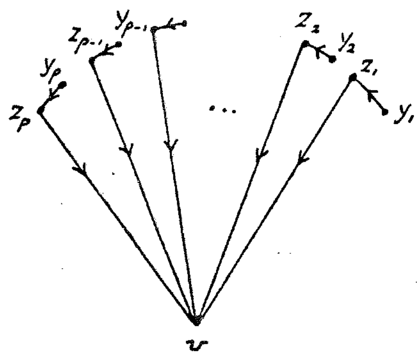

Figure 2(b)

Furthermore, let R be a graph obtained by putting c copies H together at a common vertex v as shown in Figure 3 (a). We have

$$t_r \geq \left\lceil \frac{2cp+1}{2} \right\rceil - 1 = cp$$

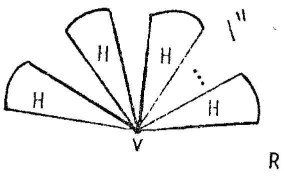

Figure 3(a)

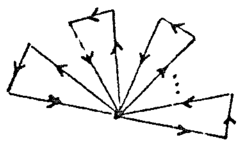

Figure 3(b)

Since it is well-known[1] that for a graph with n vertices $t_r \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, we have for the graph H in Figure 2

$$t_r = p$$

and for the graph R in Figure 3(a)

$$t_r = cp$$

Note that our result includes the result that the sequential diagnosibility of the graph R in Figure 3(b) is equal to c as was derived in [3,4].

As a generalization of the result in Theorem 1, we consider the graph B shown in Figure 4 in which there is a cycle of m vertices. At each vertex in the cycle, a copy of a graph R is attached.

We state without proof that

Theorem 2: For the graph in Figure 4

$$t_r \geq \sqrt{2(2cp+1)m} \; -2$$

4. Probabilistic Diagnosis

Many of the ideas developed here can be extended immediately to a model in which there is a given probability of failure $p(v_i)$ for each vertex v_i in the graph $G = (V, E)$[3]. We shall assume that $p(v_i) \leq \frac{1}{2}$ for all v_i. We assume that failures of the units in the system are independent. Thus, the probability of occurrence of a configuration C, denoted p(C), is computed as

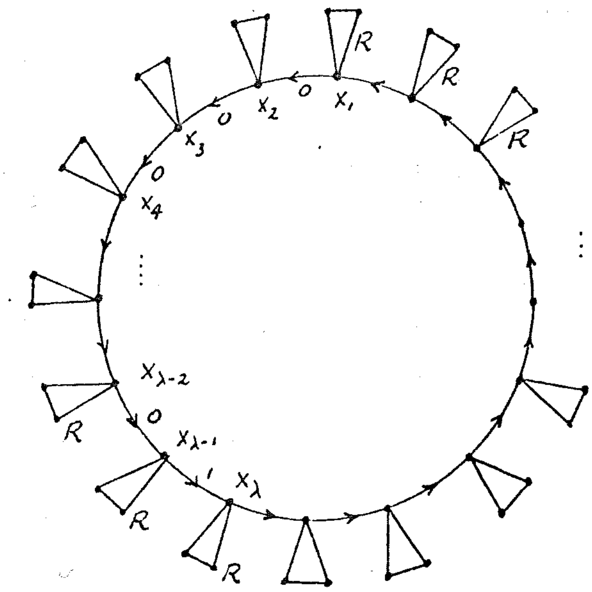

Figure 4

$$p(C) = \prod_{\substack{v_i \text{ is} \\ \text{faulty in } C}} p(v_i) \prod_{\substack{v_j \text{ is fault-} \\ \text{free in } C}} (1 - p(v_j))$$

The threshold probability for one-step diagnosis, p_0, is defined as follows: For a given syndrome S, among all the configurations that S is consistent with, atmost one of them has a probability of occurrence larger than p_0. The threshold probability for sequential diagnosis, p_r, is defined as follows: For a given syndrome S, among all the configurations that S is consistent with, let $\{C_1, C_2, \ldots, C_i\}$ denote the set of configurations with probability of occurrence larger than p_r. Then, either this set is empty or there is a unit that is faulty in all the configurations $C_1, C_2, \ldots, C_i$.

Let S be a given syndrome. For a vertex v in G, let $C_1, C_2, \ldots, C_i$ denote the configurations that S is consistent with in which v is fault-free. Let

$$P_0^S(v) = \max\left(p(C_1), p(C_2), \ldots, p(C_i)\right)$$

Let $C_1', C_2', \ldots, C_j'$ denote the configurations that S is consistent in which v is faulty. Let

$$P_1^S(v) = \max\left(p(C_1'), p(C_2'), \ldots, p(C_j')\right)$$

($P_0^S(v)$ is defined to be zero if the set $\{C_1, C_2, \ldots, C_i\}$ is empty. $P_1^S(v)$ is defined to be zero if the set $\{C_1', C_2', \ldots, C_j'\}$ is empty.) Analogous to Eq. (1), we have

$$P_r = \max_S \left[\min_{v \in V} \left[\min \left(P_0^S(v),\ P_1^S(v) \right) \right] \right] \quad (2)$$

We shall use Eq. (2) to determine upperbounds on the value of p_r. Note that in a deterministic model, it is desirable that t_r be as large as possible. Consequently, it is meaningful to determine lower bounds of t_r. In a probabilistic model, it is desirable that p_r be as small as possible. (Because we assume that the probability of failure of a unit is smaller than or equal to $\frac{1}{2}$. The smaller p_r is the more the number of configurations we admit in our consideration for a given syndrome.) Thus, analogous to Theorem 1, we have an upperbound of p_r

Theorem 3: Let G = (V, E) contain a 2-star of size k. Assume that all units in V have the same probability of failure p. Then

$$p_r \leq \sqrt{p^k (1-p)^{2|V|-k}}$$

Theorem 3 can be written in a more general form when the probabilities of failure of the units are not identical. We leave it to the reader as a simple exercise.

Application of Theorem 3 to determine upper bounds of p_r is similar to that of Theorem 1 to determine lower bounds of t_r. We thus spare ourselves with the repetition.

4. Remarks

Our results on lower bounds on t_r and upper bounds on p_r point out the desirability of having a 2-star of large size in G, which indeed provide a partial answer to the questions:

(1) Given an undirected graph G, how do we orient the edges so that the value of t_r is as large as possible (or the value of p_r is as small as possible)?

(2) Given a directed graph G how do we add new edges to G so that the value of t_r is as large as possible (or the value of p_r is as small as possible)?

According to our discussion above, we can attempt to create a 2-star that is as large as possible. As an example, we propose the following procedure for orienting the edges of an undirected graph G:

(i) Let v be the vertex with the largest degree.

(ii) For all the vertices $v_1, v_2, \ldots, v_i$ that are adjacent to v, orient the edges such that there are direct edges (v_1, v), (v_2, v), ..., (v_i, v).

(iii) Let $v_1', v_2', \ldots, v_j'$ be all the vertices that are adjacent to $v_1, v_2, \ldots, v_i$ (excluding v, and $v_1, v_2, \ldots, v_i$ themselves). Apply a maximum match algorithm to match a maximum number of vertices in $v_1', v_2', \ldots, v_j'$ to the vertices in $v_1, v_2, \ldots, v_i$. If v_a' is matched with v_b, let there be a directed edge (v_a', v_b).

(iv) Change the orientation obtained in (ii) and (iii) if necessary and orient the remaining edges in the graph in such a way that the resultant graph is strongly connected. Note that in this way, we obtain a largest possible 2-star with its root at v.

In conclusion, we note that we obtain a general result that enables us to determine lower bounds of the sequential diagnosibility t_r and upper bounds of the threshold probability of sequential diagnosibility p_r of digital systems. In view of the difficulty of determining the precise values of t_r and p_r, we believe that our results provide a reasonable and practical approach to solve the problem in exactly the same spirit as using approximation algorithms to solve NP-complete problems.

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