ANELASTIC DEFORMATION AS AN INTERNAL PARAMETER

G. Capriz - L.M. Saha

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Istituto di Elaborazione della Informazione, C.N.R. - Pisa

1. Introduction

The suggestion of Coleman and Noll [1] that the Clausius-Duhem inequality could be used to impose restrictions upon the choice of constitutive relations (and even to indicate limits of acceptability of the criterion of equipresence) has been followed so many times that a new paper exploiting that device needs to be justified: we plead that our interpretation leading to a property of alternatives is a significant variant of the well-known argument.

The class of continua we are thus led to consider can be thought of as a special case of continua with internal parameters, and a certain theorems of Coleman and Gurtin [2] can be adapted to advantage.

2. Thermodynamic processes, constitutive equations.

We accept the notation and the hypotheses of Ref. [1] (1) with these changes:

(1) And we do not recall then here for brevity, because they largely standard and also because Coleman and Noll's paper is now even more generally available in Noll's Selecta. [3]

(i) the description of a thermodynamic process requires nine fields (rather than eight), we ninth field being the invertible tensor $\underline{M}(\underline{X}, t)$ which can be thought to define in the sense of Noll [4, Sect. 3] a local placement consequence of an anelastic deformation.

(ii) the constitutive equations are

$$\psi = \hat{\psi}(\underline{M}, \underline{N}, \vartheta, \mathrm{grad}\,\vartheta), \quad \eta = \hat{\eta}(\underline{M}, \underline{N}, \vartheta, \mathrm{grad}\,\vartheta),$$

$$\underline{T} = \hat{\underline{T}}(\underline{M}, \underline{N}, \vartheta, \mathrm{grad}\,\vartheta), \quad \underline{q} = \hat{\underline{q}}(\underline{M}, \underline{N}, \vartheta, \mathrm{grad}\,\vartheta). \tag{2.1}$$

where we have put $\underline{N} = \underline{F}\,\underline{M}^{-1}$.

We invoke now the Clausius-Duhem inequality

$$\rho(\dot{\psi} + \eta\dot{\vartheta}) - \mathrm{tr}\{\underline{T}\,\underline{L}\} + \frac{1}{\vartheta}\underline{q}\cdot\mathrm{grad}\,\vartheta \leq 0,$$

which, in view of the constitutive equations assumed above and of the fact that the values of $\dot{\vartheta}$ and $\overline{\mathrm{grad}\,\vartheta}^{\,\cdot}$ can be locally arbitrary, exclude a dependence of $\hat{\psi}$ and $\hat{\eta}$ on $\mathrm{grad}\,\vartheta$ and impose the entropy relation. Furthermore we have (with the convention of Ref [1] regarding the meaning of $\hat{\psi}_{\underline{N}}$, $\hat{\psi}_{\underline{M}}$):

$$\mathrm{tr}\{(\rho\hat{\psi}_{\underline{N}} - \underline{N}^{-1}\hat{\underline{T}})\dot{\underline{N}}\} + \mathrm{tr}\{(\rho\hat{\psi}_{\underline{M}} - \underline{F}^{-1}\hat{\underline{T}}\,\underline{N})\,\dot{\underline{M}}\} + \frac{1}{\vartheta}\underline{q}\cdot\mathrm{grad}\,\vartheta \leq 0 \tag{2.2}$$

and this residual inequality confronts us with the alternatives:

(i) Both $\underline{N}$ and $\underline{M}$ can be chosen arbitrarily and indepedently (on a particle at all times); then the decomposition of $\underline{F}$ into the product $\underline{N}\underline{M}$ is almost irrelevant for our present purposes (as it might, at most, influence $\underline{q}$), because $\hat{\psi}$ must satisfy the partial differential equation

$$\underline{M}\,\hat{\psi}_{\underline{M}} = \hat{\psi}_{\underline{N}}\,\underline{N}$$

and hence must depend on the product $\underline{F} = \underline{N}\,\underline{M}$ and not on $\underline{N}$ and $\underline{M}$ separately; furthermore the usual stress relation follows.

(ii) Only $\underline{N}$ (for instance) can be chosen arbitrarily and independently, whereas $\underline{M}$ depends on the choice of the other fields entering inequality (2.1), but not on the choice of $\mathring{\underline{N}}$. Then

$$\hat{\underline{T}} = \rho\,\underline{N}\,\hat{\psi}_{\underline{N}}\,, \quad (2) \qquad (2.3)$$

whereas $\underline{M}$ develops necessarily as an internal parameter, following a differential law; to state an acceptable form of such a law we must discuss first questions of objectivity.

(iii) $\underline{N}$ and $\underline{M}$ are interrelated.

3. Objectivity.

We pursue from now on the second alternative; this choice lifts already, at least in part, our considerations from the initial purely formal setting. Another substantial decision regards the influence upon $\underline{M}$ and $\underline{N}$ of a change of observer, specified by an orthogonal tensor $\underline{Q}$. In view of the significance we want to attribute to $\underline{M}$ we will presume

$$\underline{M} \rightarrow \underline{Q}\,\underline{M} \quad \text{and consequently} \quad \underline{N} \rightarrow \underline{Q}\,\underline{N}\,\underline{Q}^{T}.$$

Objectivity then leads to reduced forms of the constitutive equations and restricts also the choice of the relation between $\dot{\underline{M}}$ and $\underline{M}$, $\underline{N}$, ϑ, grad ϑ, because $\underline{M}$ cannot appear alone but only through the combination $\frac{d}{dt}(\underline{M}^{T}\underline{M})$. Particularly significant

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(2) If we introduce Noll's stress tensor $\underline{T}_N = |\underline{N}|\,\underline{T}(\underline{N}^{-1})^{T}$ (see [4], Sect. 15) this equation reads $\underline{T}_N^{T} = \rho|\underline{N}|\hat{\psi}_{\underline{N}}$.

seems to be the case when the latter relation can be made explicit, as follows

$$\dot{\underline{E}}^{(M)} = \underline{G}(\underline{E}^{(M)}, \underline{E}, \nabla\vartheta, \vartheta) \tag{3.2}$$

where:

$$\underline{E} = \frac{1}{2}(\underline{F}^T\underline{F}-\underline{1}) \quad , \; \underline{E}^{(M)} = \frac{1}{2}(\underline{M}^T\underline{M}-\underline{1}) \quad , \quad \nabla\vartheta = \underline{F}^T \operatorname{grad}\vartheta \; ,$$

and the right-hand side is so regular that a unique solution exists of any initial value problem in $\underline{E}^{(M)}$ over an appropriate interval (t_o, t) once $\underline{N}(t)$, $\vartheta(t)$, $\nabla\vartheta(t)$ are given in (t_o, t).

We may remark that (3.1) does not fully determine the development of $\underline{M}$; in fact the variables in the right-hand side are chosen so as to be unaffected if $\underline{N}$ and $\underline{M}$ are changed into $\underline{NR}^T$ and $\underline{RM}$ respectively, where $\underline{R}$ is an orthogonal tensor.

4. Anelastic deformation as an internal parameter.

The relations (2.1), (2.2) mirror the setting presupposed by Coleman and Gurtin in Ref. [2] with $\underline{E}^{(M)}$ taking the place of the parameter $\underline{\alpha}$ (3), we can therefore adapt now some of their definitions.

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(3) Notice that $\underline{E}^{(M)}$, as $\underline{\alpha}$, is not affected by a change of observer; notice also that the explicit assumption (4.5) in Ref. [2] on the internal parameter $\underline{\alpha}$ need not have been made as it is largely implied by the hypothesis that ψ depends on $\underline{\alpha}$ and by the Clausius-Duhem inequality (either $\dot{\underline{\alpha}}$ depends on the other fields or ψ is independent of $\underline{\alpha}$).

We call equilibrium state $\mathcal{S}^*$ for a body at a material point a triplet ($\underline{M}^*$, $\underline{N}^*$, ϑ^*) such that $\underline{G}$ vanishes when ($\underline{M}$, $\underline{N}$, ϑ) are specified as ($\underline{M}^*$, $\underline{N}^*$, ϑ^*) and $\nabla\vartheta$ is put equal to zero.

We define the domain of attraction of $\mathcal{S}^*$ at constant total strain $\underline{F}^* = \underline{N}^* \underline{M}^*$ and constant temperature ϑ^* as the set $D(\mathcal{S}^*)$ of all $\underline{M}^\circ$ such that the solution of the initial value problem

$$\dot{\underline{E}}^{(M)} = \underline{G}\left[\underline{E}^{(M)}, \frac{1}{2}(\underline{F}^{*T}\underline{F}^* - \underline{1}), \underline{0}, \vartheta^*\right],$$

$$\underline{E}^{(M)}(0) = \frac{1}{2}(\underline{M}^{\circ T}\underline{M}^\circ - \underline{1}),$$

exists for all t and tends to $\frac{1}{2}(\underline{M}^{*T}\underline{M}^* - \underline{1})$.

Correspondingly we can introduce the concept of asymptotic stability. As an example of these definitions and of the adaptation of Coleman and Noll reasoning we quote the following theorem: if $\mathcal{S}^*$ is an asymptotically stable equilibrium state at constant total strain and constant temperature, then

$$\hat{\psi}(\underline{M}, \underline{F}^*\underline{M}^{-1}, \vartheta^*) \geq \hat{\psi}(\underline{M}^*, \underline{N}^*, \vartheta^*),$$

for all $\underline{M}$ in some neighbourhood of $\underline{M}^*$. Furthermore, when $\hat{\psi}$ does not depend on $\underline{M}$, $\hat{\psi}_{\underline{N}}$ (and therefore the stress) vanishes at $\mathcal{S}^*$.

References

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[2] B.D. Coleman, M.E. Gurtin, Thermodynamics with internal state variables. J. Chem. Phys., 47 (1967), 597-613.

[3] W. Noll, The foundations of mechanics and thermodynamics. Springer, Berlin (1974).

[4] W. Noll, Materially uniform simple bodies with inhomogeneities. Arch.Rat. Mech. Anal., 27 (1967), 1-32.